ERROR REDUCTION USING ADAPTIVE MONTE CARLO

THE REDUCED SOURCE METHOD

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http://www-xdiv.lanl.gov/XTM/projects/mc21/docs/

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THE REDUCED SOURCE METHOD

- INTRODUCTION
- THEORETICAL DERIVATION
- APPLICATION TO A 1-D PROBLEM
- CONVERGENCE RESULTS
- FUTURE EFFORT

INTRODUCTION

- **Discrete sequential Monte Carlo**

  - Halton (1962), Proc. Camb. Phil. Soc. **58**, 57-78

  - Halton (1967), Univ. Wisconsin, Madison, MRC 816

  - Halton (1994), J. Sci. Comput. **9**(2), 213-257

    Considered linear discrete systems
    Considered four infinite-series estimators
    Proved exponential convergence exists

$$\frac{V^n}{V^0} \le C^n \qquad (C < 1)$$

INTRODUCTION

- **Continuous reduced source method**

- Fraley et al. (1974), Physics of Fluids **17**(2), 474-489

  Photon transport
  Deterministic solution for the initial guess
  Monte Carlo solution for the difference (reduced source)

- Booth (1984), Los Alamos National Lab., LA-10465-PR, 43-49

  Outlined the iteration scheme
  Suggested the use of discretization for source reduction
  Suggested the use of flux expansion for source reduction

THEORETICAL DERIVATION

- Transport equation
- Iteration approach
- Simplifications
- Approximations

TRANSPORT EQUATION

- **In terms of the angular flux (φ):**

$$\frac{1}{\upsilon}\frac{\partial\varphi}{\partial t} + \hat{\Omega} \bullet \nabla\varphi + \Sigma_t(r, E)\varphi(r, E, \hat{\Omega}, t)$$

$$= \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \Sigma_s(E' \to E, \hat{\Omega}' \to \Omega)\varphi(r, E', \hat{\Omega}', t) + Q(r, E, \hat{\Omega}, t)$$

- **Homogeneous, steady-state**

$$\hat{\Omega} \bullet \nabla\varphi + \Sigma_t(E)\varphi(r, E, \hat{\Omega}) = \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \Sigma_s(E' \to E, \hat{\Omega}' \to \Omega)\varphi(r, E', \hat{\Omega}') + Q(r, E, \hat{\Omega})$$

ITERATION APPROACH

- **Let φ = M + D**

$$\hat{\Omega} \bullet \nabla D + \Sigma_t(E) D(r, E, \hat{\Omega}) = \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \Sigma_s(E' \to E, \hat{\Omega}' \to \Omega) D(r, E', \hat{\Omega}') + S(r, E, \hat{\Omega})$$

$$S(r, E, \hat{\Omega}) = Q + \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \Sigma_s(E' \to E, \hat{\Omega}' \to \Omega) M(r, E', \hat{\Omega}') - \hat{\Omega} \bullet \nabla M - \Sigma_t(E) M$$

- **Introduce iteration index n=0,1,2,...**

Pick M^0

$$S^n = Q + \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \Sigma_s(E' \to E, \hat{\Omega}' \to \Omega) M^n - \hat{\Omega} \bullet \nabla M^n - \Sigma_t(E) M^n$$

$$\hat{\Omega} \bullet \nabla D^n + \Sigma_t(E) D^n = \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \Sigma_s(E' \to E, \hat{\Omega}' \to \Omega) D^n + S^n$$

$$M^{n+1} = M^n + D^n$$

n=n+1

SIMPLIFICATIONS

- **One speed, 1-D, one direction, $\Sigma_t = \Sigma_a$**

Pick $M^0(x)$

$$S^n(x) = Q(x) - \frac{d}{dx}M^n(x) - \Sigma_t M^n(x)$$

$$\frac{d}{dx}D^n(x) + \Sigma_t D^n(x) = S^n(x)$$

$$M^{n+1}(x) = M^n(x) + D^n(x)$$

n=n+1

SIMPLIFICATIONS

- **Our implementation**

Pick $M^0=0$, giving $S^0(x)=Q(x)$

$$\frac{d}{dx}D^n(x) + \Sigma_t D^n(x) = S^n(x)$$ Solve for D^n using Monte Carlo

$$M^{n+1}(x) = M^n(x) + D^n(x)$$ Update flux estimate

$$S^{n+1}(x) = Q(x) + F\{M^{n+1}(x)\}$$ Reduce source

$$R^n(x) = D^n(x)/M^{n+1}(x)$$ Compute relative errors

n=n+1

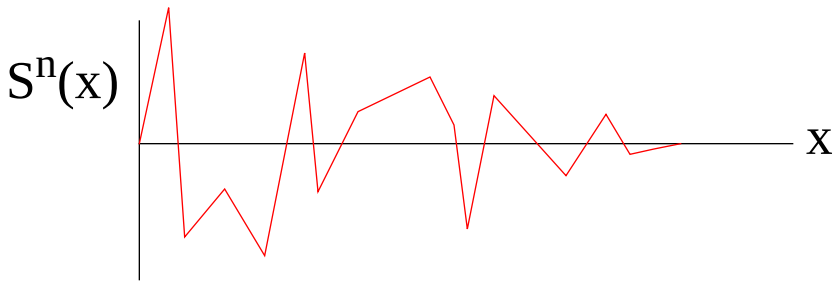
- **Functional poses difficulty**

$$F\{M^n(x)\} \equiv -\left[\frac{d}{dx}M^n(x) + \Sigma_t M^n(x)\right]$$

APPROXIMATIONS

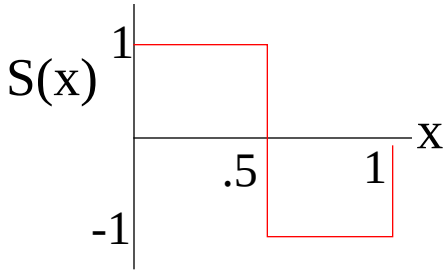
- **Source density fluctuations**

  - $S^n(x)$ can be "randomly" positive and negative



  - Monte Carlo source requires integral of $S^n(x)$: $\pm w^n = \int \left| S^n(x) \right| dx$

  - Care must be taken in performing this integral

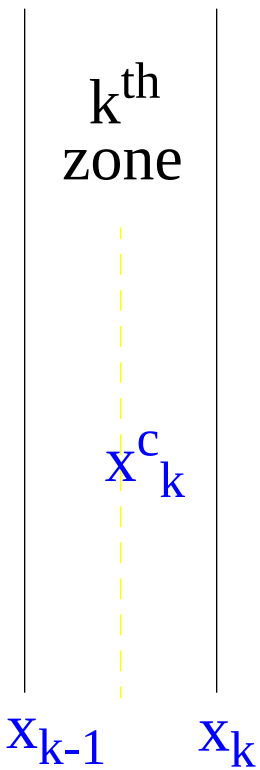


$$w = \int S(x)\,dx = \left(1 \cdot \frac{1}{2}\right) + \left(-1 \cdot \frac{1}{2}\right) = 0 \qquad \text{no}$$

$$w = \int |S(x)|\,dx = \begin{pmatrix} 1 & 0<x<.5 \\ -1 & .5<x<1 \end{pmatrix} \qquad \text{yes}$$

APPROXIMATIONS

- **Source density discretization**



- Tally $M^n(x)$ at the center and boundary of each zone

- Compute source density at the center of the kth zone:

$$S_k^n = F_k\{M^n(x)\} = -\left[\frac{M^n(x_k) - M^n(x_{k-1})}{\Delta x} + \Sigma_a M^n(x_k^c)\right]$$

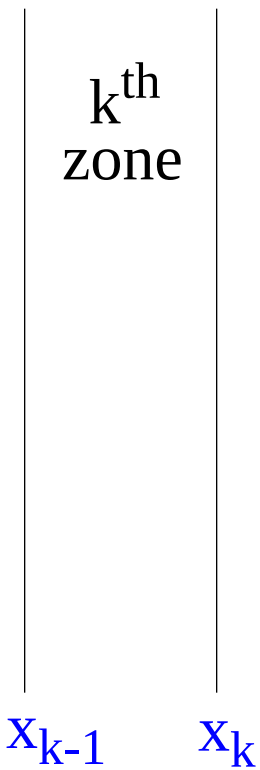
- Assume $S^n(x)$ is uniform across each zone

- Monte Carlo source weight becomes (K=# zones)

$$\pm w^n = \sum_{k=1}^{K} \left| S_k^n \Delta x \right|$$

APPROXIMATIONS

- **Integral source discretization**



- Tally $M^n(x)$ at the boundary of each zone

- Compute the integral for the kth zone:

$$I_k^n = \int_{x_{k-1}}^{x_k} F\{M^n(x)\}dx = -\int_{x_{k-1}}^{x_k} \left[\frac{dM^n(x)}{dx} + \Sigma_a M^n(x)\right]dx$$

$$= M^n(x_{k-1}) - M^n(x_k) - \Sigma_a \int_{x_{k-1}}^{x_k} M^n(x)dx$$

- Estimate the integral of $M^n(x)$ by Simpson's rule

- Monte Carlo source weight becomes (K=# zones)

$$\pm w^n = \sum_{k=1}^{K} \left|I_k^n\right|$$

APPROXIMATIONS

- **Sampling of the source (n>0)**

  - Source zone is sampled from a histogram

$$\xi_1 = \frac{1}{w^n} \sum_{j=1}^{k} \left| S_j^n \Delta x \right| \quad \text{or} \quad \xi_1 = \frac{1}{w^n} \sum_{j=1}^{k} \left| I_j^n \right|$$

  - Source position is sampled uniform in zone k

$$x = x_{k-1} + \xi_2 \Delta x$$

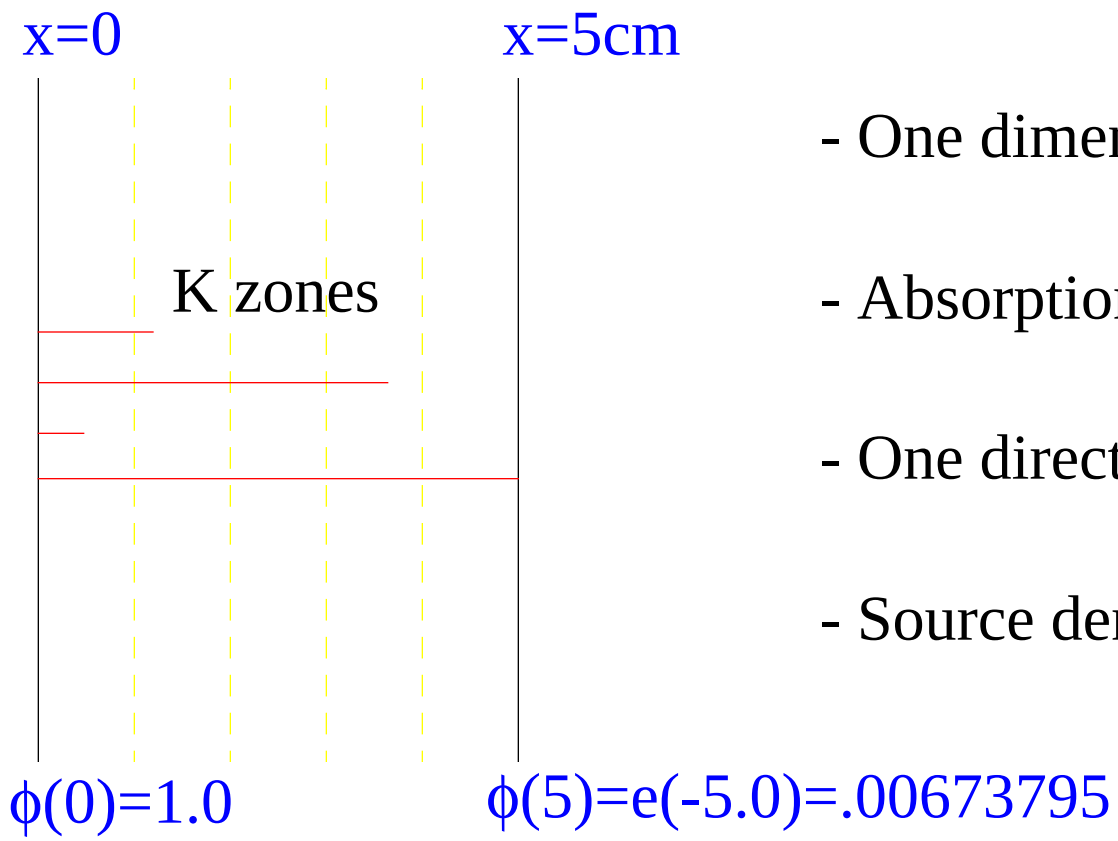
- **Accuracy of the discretization**

  - Accuracy of M(x) will be limited by the accuracy of w^n
  - As n>>1, estimated error may become much less than true error
  - Expect to encounter “FALSE CONVERGENCE”

APPLICATION TO A 1-D PROBLEM

- Description
- Overview
- First iteration
- Second iteration
- Third iteration

DESCRIPTION



- One dimension (x)
- Absorption only ($\Sigma_t = \Sigma_a = 1\text{cm}^{-1}$)
- One direction (μ=1)
- Source density, $Q(x) = \delta(x-0)$

OVERVIEW

- **Monte Carlo method**

  - Source weight,

$$w^0 = \int Q(x)dx = \int_{-\infty}^{\infty} \delta(x-0)dx = 1 \qquad \text{for n=0 (1}^{\text{st}}\text{ iter.)}$$

$$\pm w^n = \sum_{k=1}^{K} \left| I_k^n \right| \qquad \text{for n>0}$$

  - Source position,

$$x = 0 \qquad \text{for n=0 (1}^{\text{st}}\text{ iter.)}$$

$$x = x_{k-1} + \xi_2 \Delta x \quad \text{where} \quad \xi_1 = \frac{1}{w^n} \sum_{j=1}^{k} \left| I_j^n \right| \qquad \text{for n>0}$$

  - Track length, $\lambda = -\ln(\xi)/\Sigma_a$

  - Particle flux, $D^n(x) = \frac{1}{N} \sum_{i=1}^{N} w_i^n$

OVERVIEW

- **Reduced source method**

- Solution for $D^0(x)$ when $M^0=0$ [i.e., $S^0(x)=Q(x)=\delta(x-0)$]

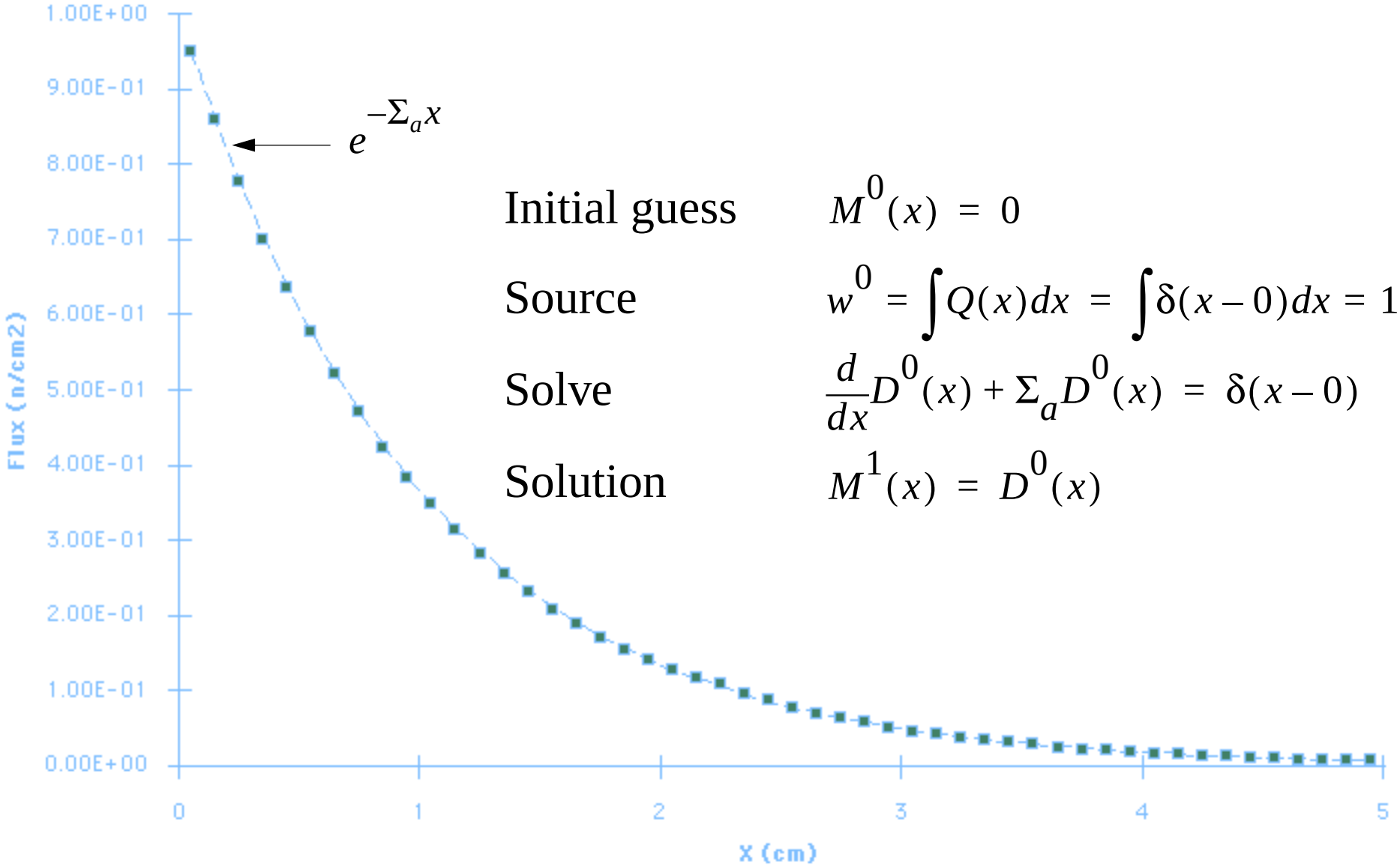
$$\frac{d}{dx}D^0(x) + \Sigma_a D^0(x) = \delta(x-0) \qquad \Rightarrow \qquad D^0(x) = e^{-\Sigma_a x}$$

- Update $M^1(x)=M^0(x)+D^0(x)$ and solve for $S^1(x)$

$$S^1(x) = Q(x) + F\{M^1(x)\} = \delta(x-0) - \left[\frac{d}{dx}D^0(x) + \Sigma_a D^0(x)\right]$$

$$= \delta(x-0) - \delta(x-0) = 0$$

FIRST ITERATION



Initial guess $\quad M^0(x) = 0$

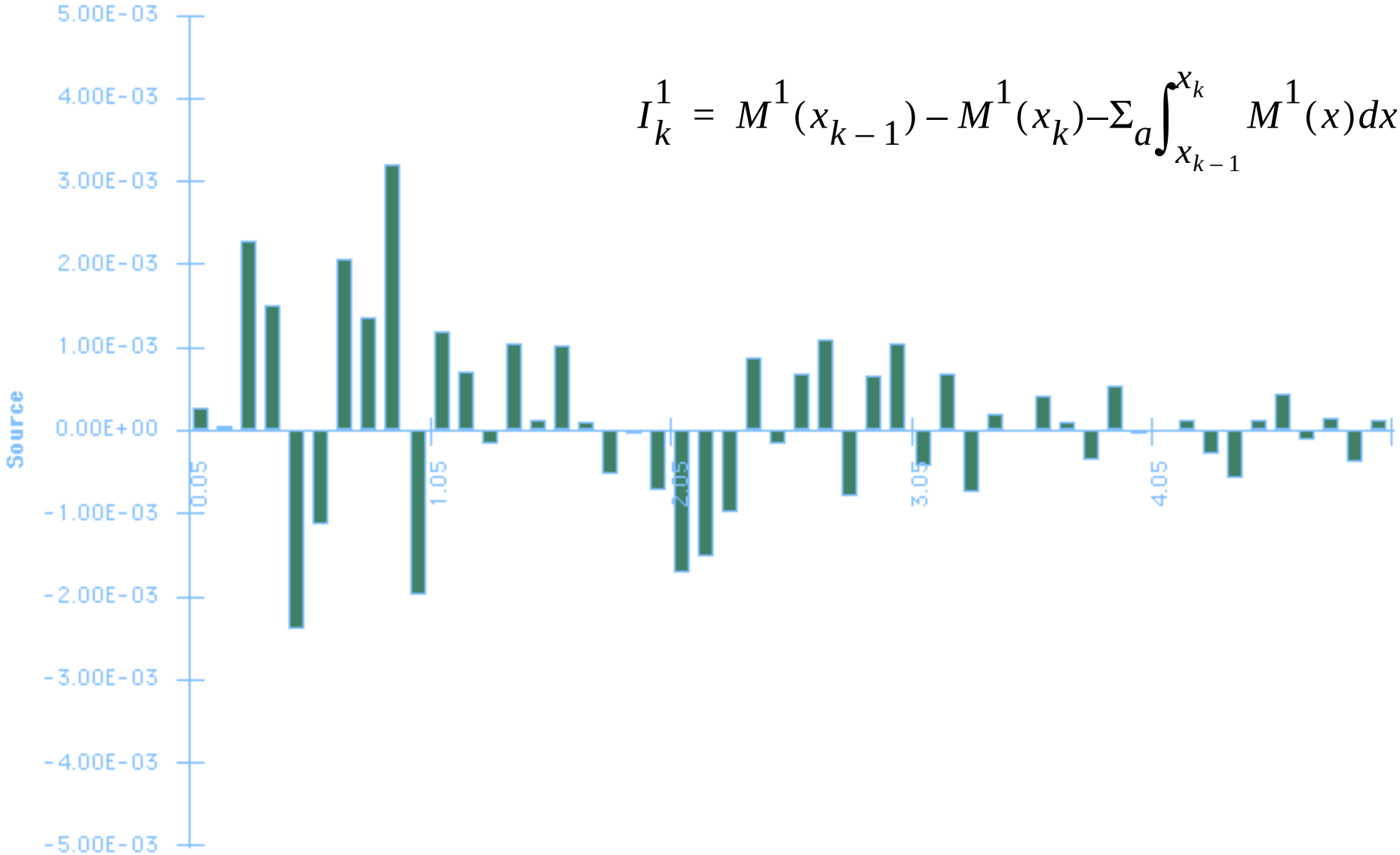
Source $\quad w^0 = \int Q(x)dx = \int \delta(x-0)dx = 1$

Solve $\quad \frac{d}{dx}D^0(x) + \Sigma_a D^0(x) = \delta(x-0)$

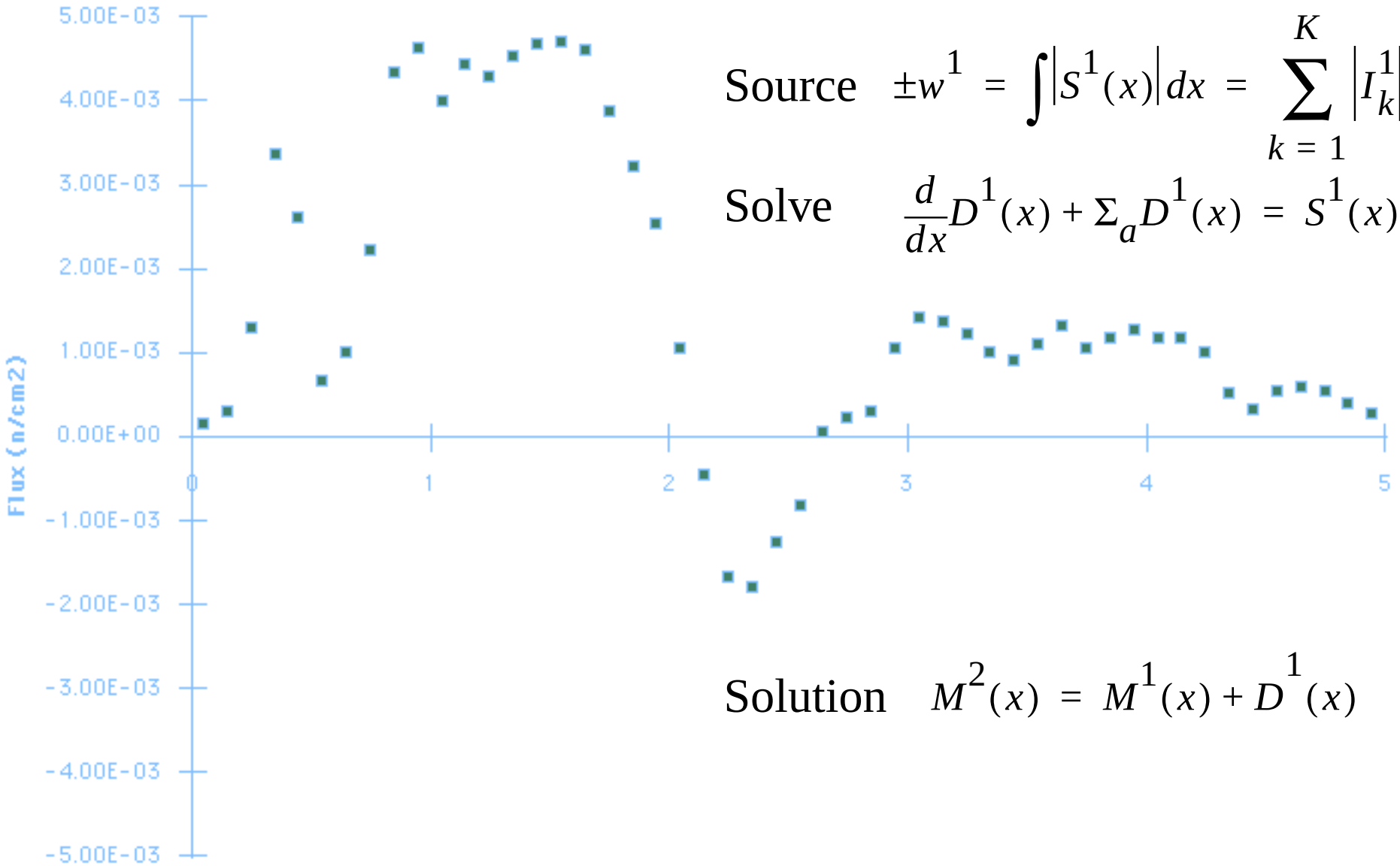
Solution $\quad M^1(x) = D^0(x)$

FIRST ITERATION

$$I_k^1 = M^1(x_{k-1}) - M^1(x_k) - \Sigma_a \int_{x_{k-1}}^{x_k} M^1(x)\,dx$$



SECOND ITERATION



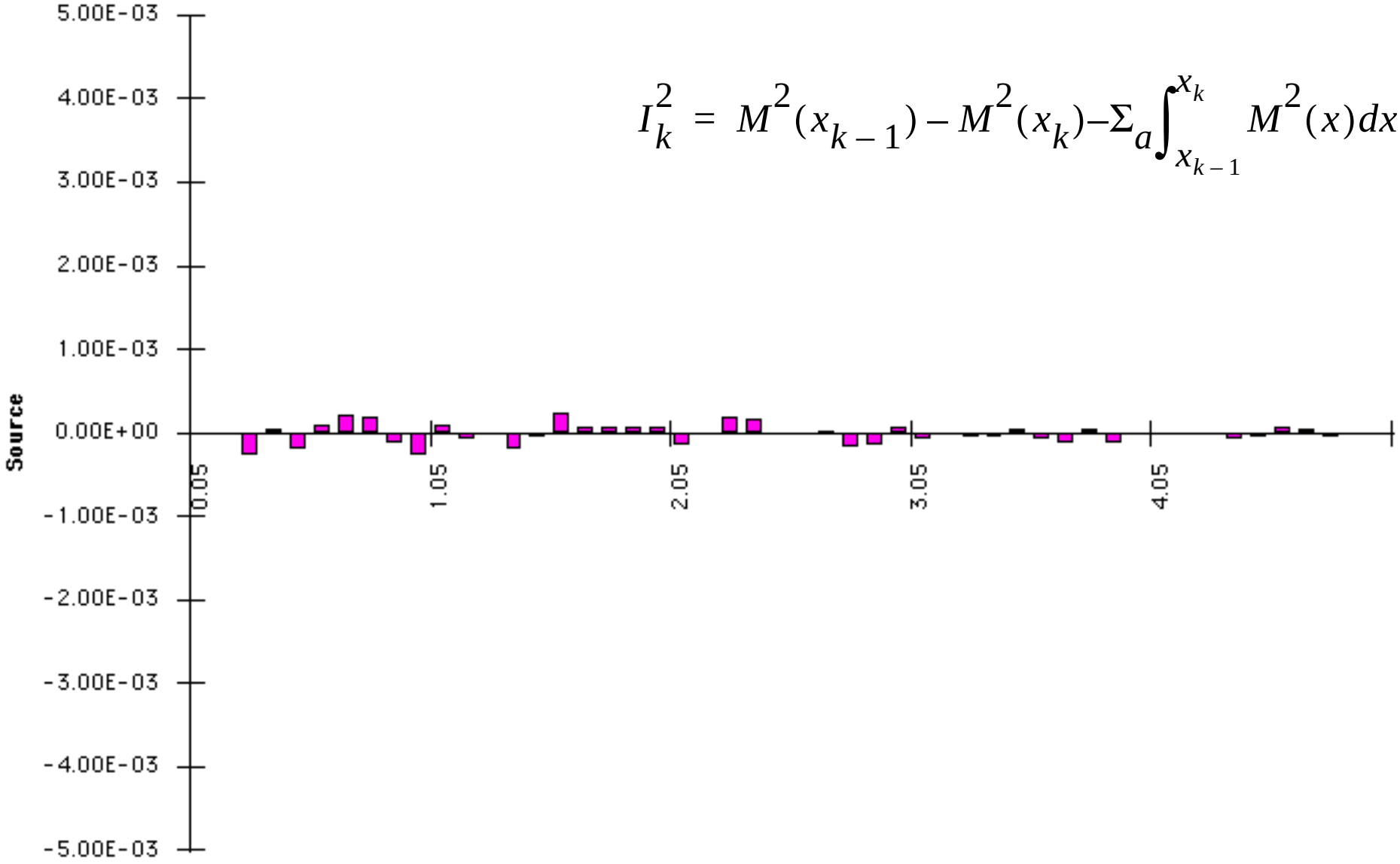
Source $\pm w^1 = \int \left| S^1(x) \right| dx = \sum_{k=1}^{K} \left| I_k^1 \right|$

Solve $\frac{d}{dx} D^1(x) + \Sigma_a D^1(x) = S^1(x)$

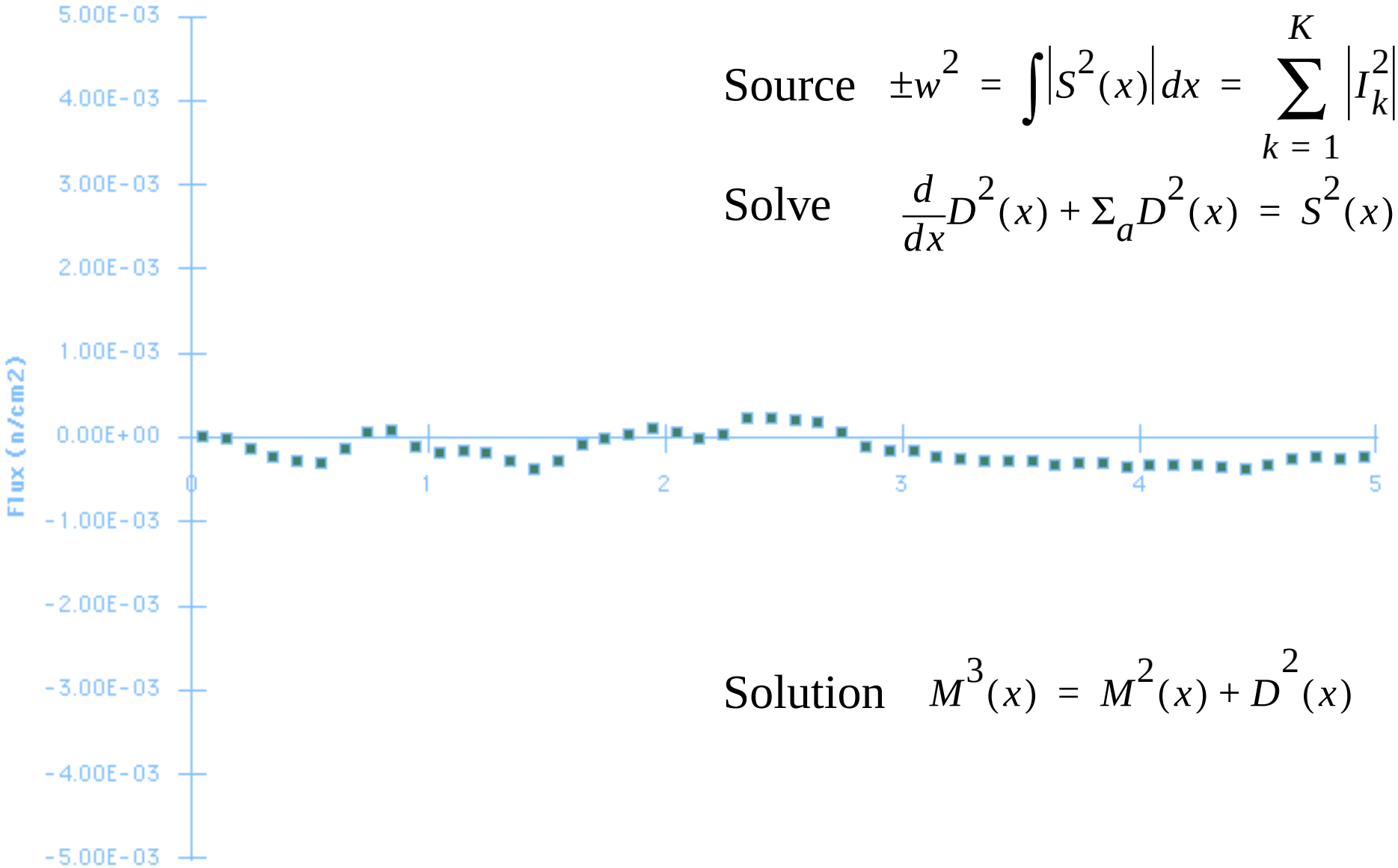
Solution $M^2(x) = M^1(x) + D^1(x)$

SECOND ITERATION

$$I_k^2 = M^2(x_{k-1}) - M^2(x_k) - \Sigma_a \int_{x_{k-1}}^{x_k} M^2(x)\,dx$$



THIRD ITERATION

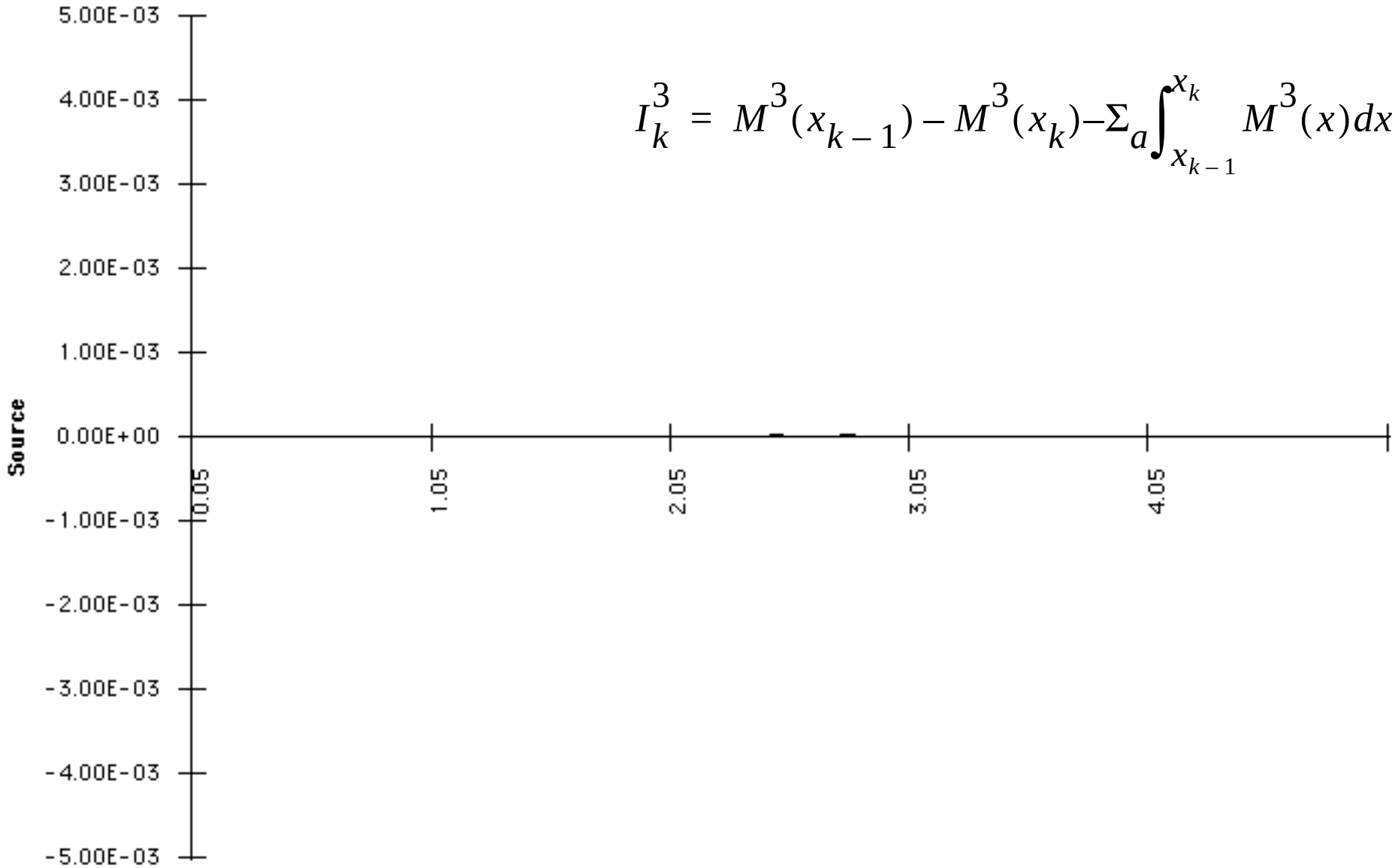


Source $\pm w^2 = \int \left| S^2(x) \right| dx = \sum_{k=1}^{K} \left| I_k^2 \right|$

Solve $\frac{d}{dx} D^2(x) + \Sigma_a D^2(x) = S^2(x)$

Solution $M^3(x) = M^2(x) + D^2(x)$

THIRD ITERATION

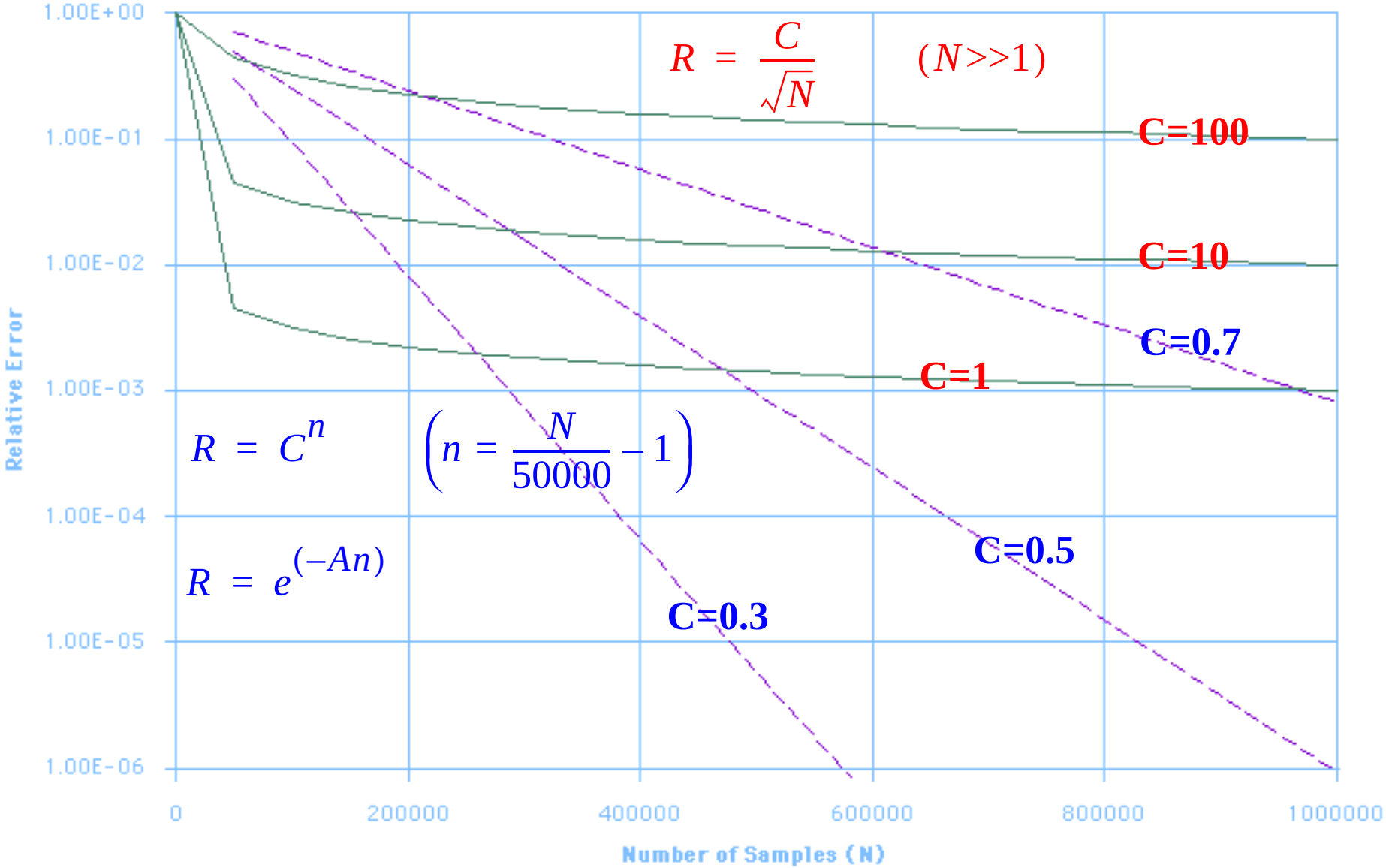


$$I_k^3 = M^3(x_{k-1}) - M^3(x_k) - \Sigma_a \int_{x_{k-1}}^{x_k} M^3(x)\,dx$$

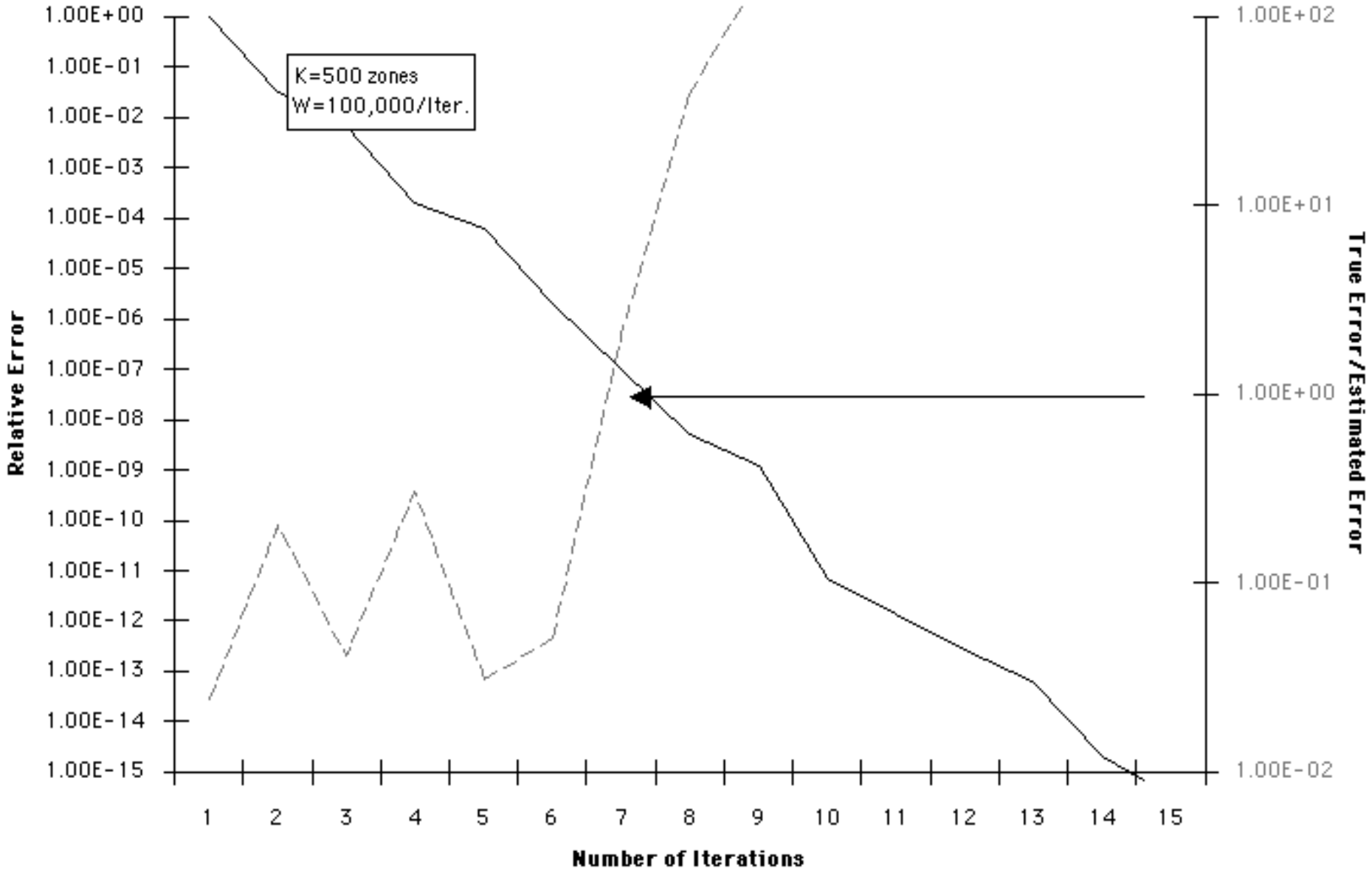
CONVERGENCE RATES

- Nominal versus exponential
- False convergence
- Varying the number of zones
- Varying the number of particles
- Optimal convergence

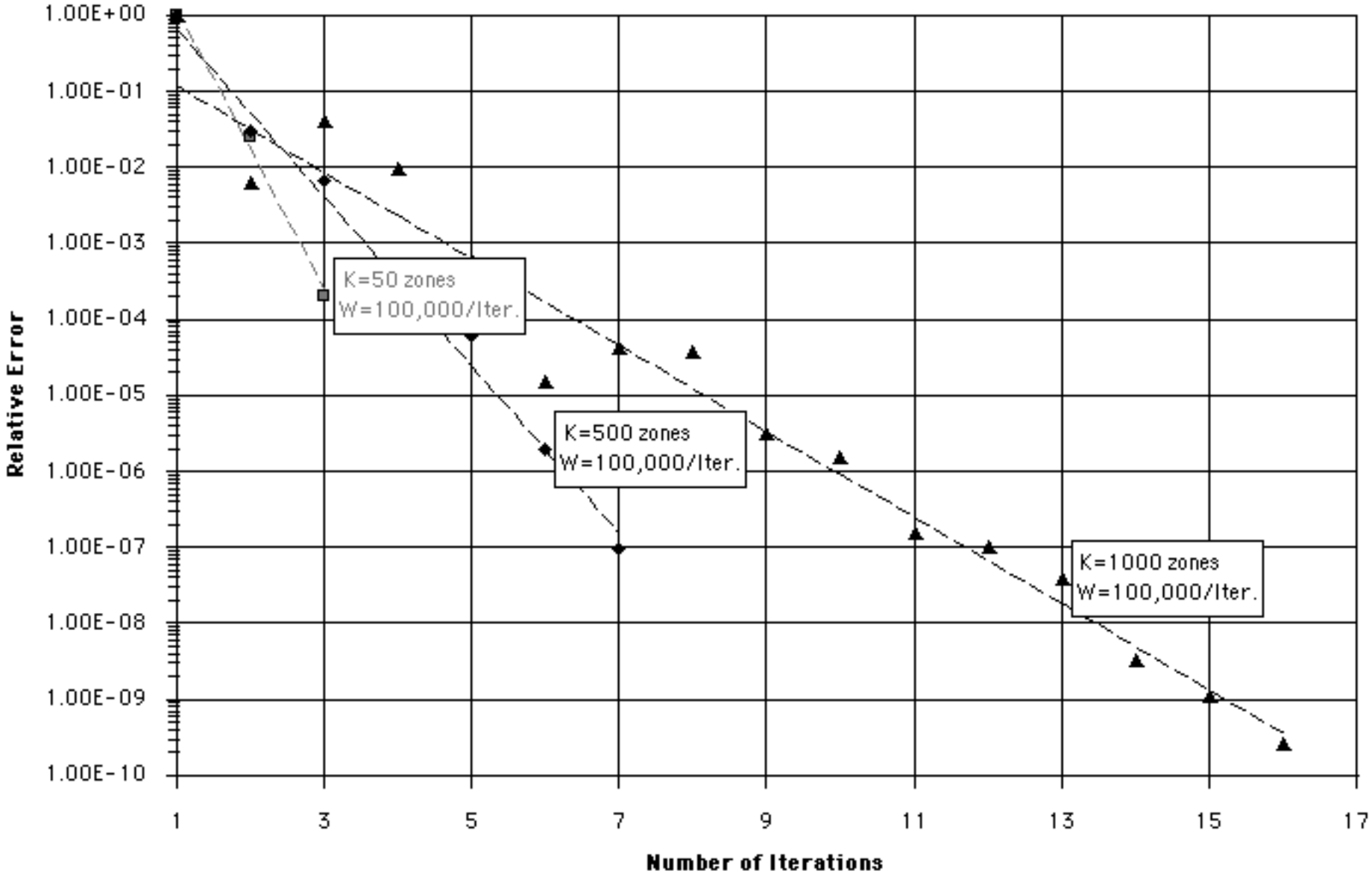
NOMINAL VERSUS EXPONENTIAL



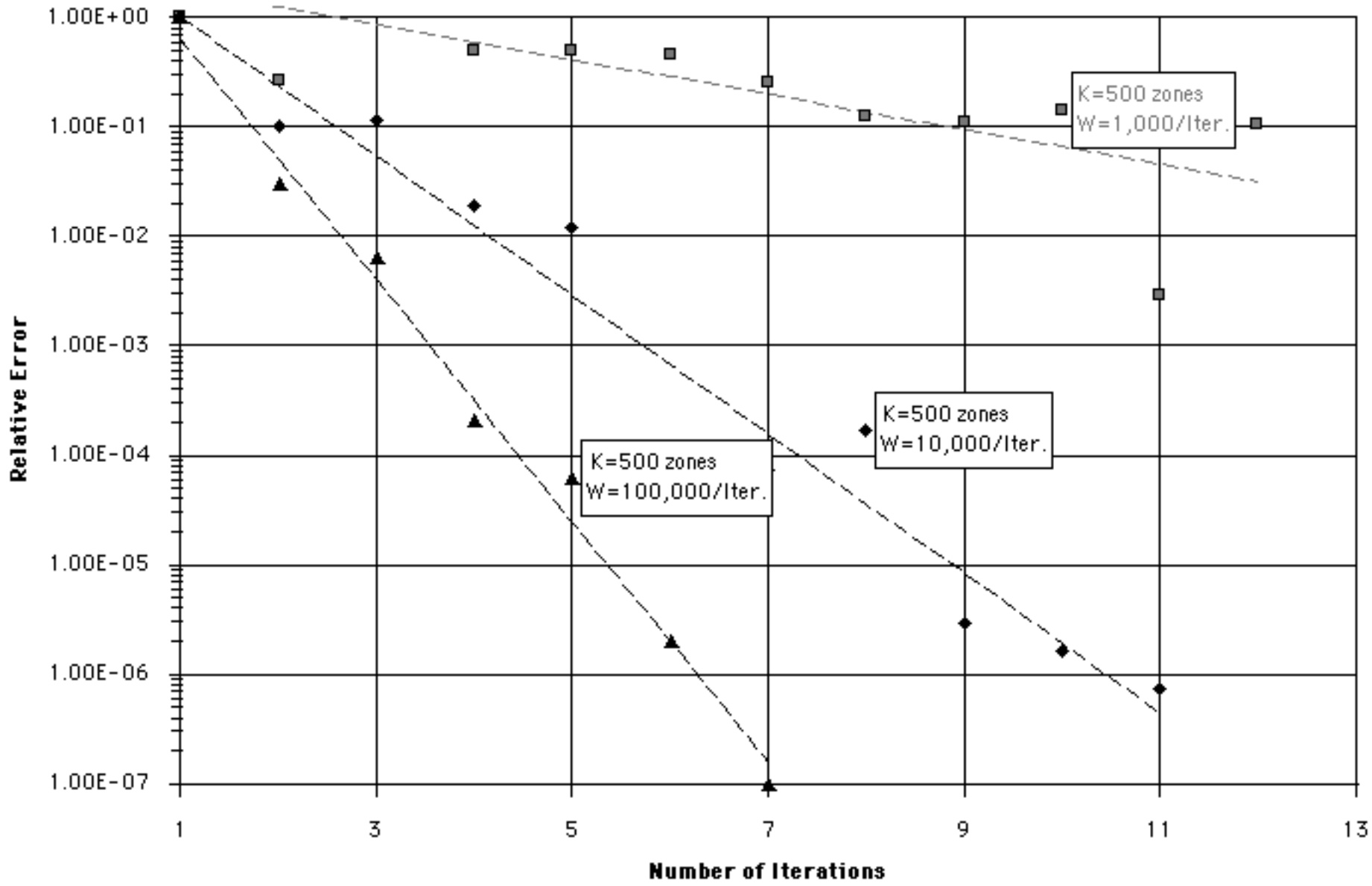
FALSE CONVERGENCE



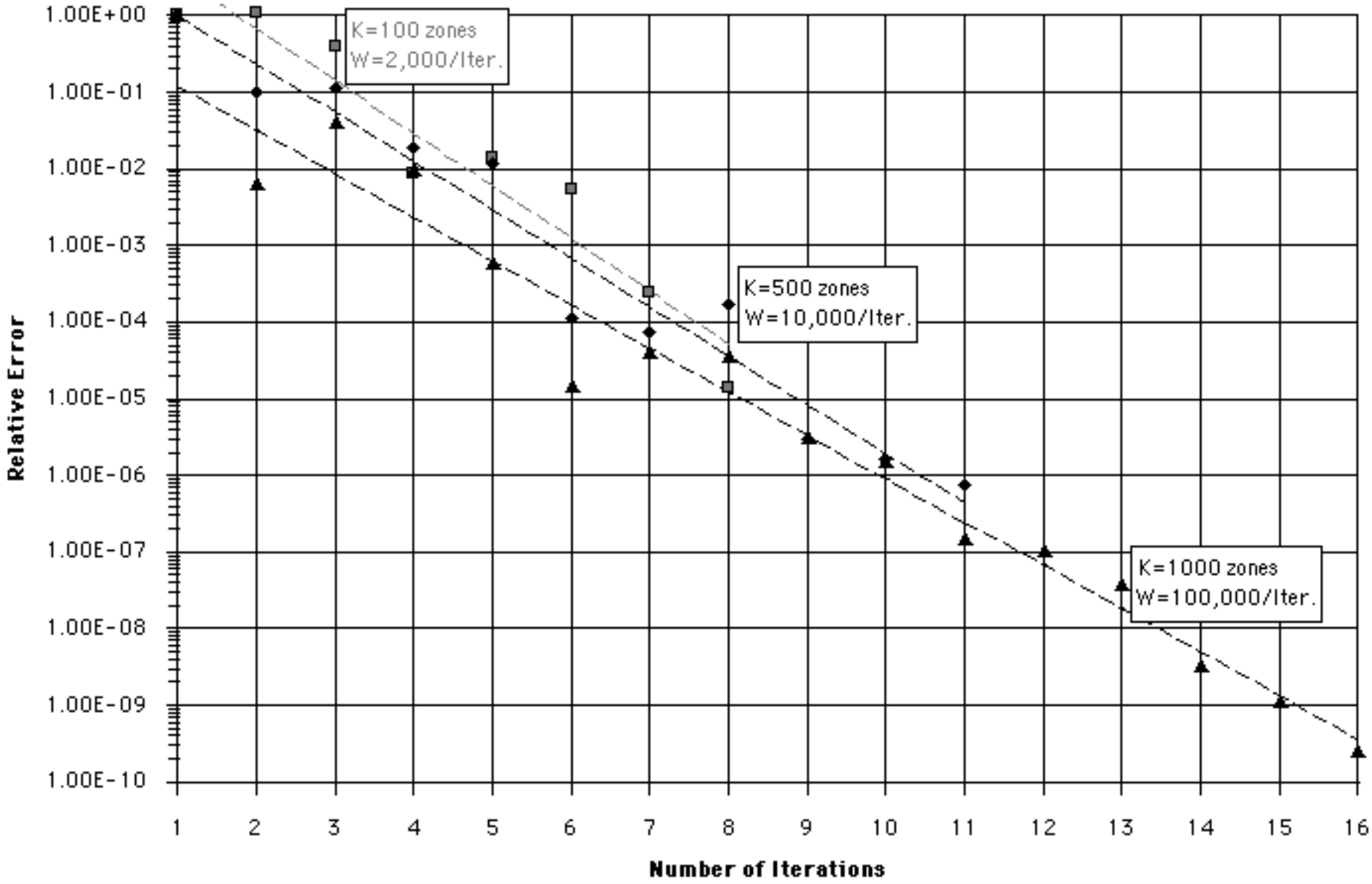
VARYING THE NUMBER OF ZONES



VARYING THE NUMBER OF PARTICLES



OPTIMAL CONVERGENCE



FUTURE EFFORT

- Legendre expansions
- 1-D bidirectional problem
- 1-D tridirectional problem
- 3-D extensions
- 3-D problem

LEGENDRE EXPANSIONS

- **Expand $D^n(x)$**

$$D^n(x) = \sum_{j=0}^{\infty} a_j^n P_j\left(\frac{2x}{T} - 1\right)$$

- **Estimate the coefficients**

$$a_j^n \approx \frac{1}{N} \sum_{i=1}^{N} \left[w^n\left(\frac{2j+1}{T}\right) \int_x^{x+\lambda} P_j\left(\frac{2y}{T} - 1\right) dy \right]$$

- **Update the solution**

$$M^n(x) = \sum_{j=0}^{\infty} \left[\sum_{m=0}^{n} a_j^m \right] P_j\left(\frac{2x}{T} - 1\right)$$

- **Reduce the source**

$$S^n(x) = -\left[\frac{d}{dx} M^n(x) + \Sigma_a M^n(x)\right]$$