Title: Analytical Benchmark Test Set for Criticality Code Verification

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Analytical Benchmark Test Set For Criticality Code Verification

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Abstract

A number of published numerical solutions to analytic eigenvalue ($k_{eff}$) and eigenfunction equations are summarized for the purpose of creating a criticality verification benchmark test set. The 75-problem test set allows the user to verify the correctness of a criticality code for infinite medium and simple geometries in one-, two-, three-, and six-energy groups, with one-, two-, and four-media. The problems include both isotropic and linearly and quadratically anisotropic neutron scattering. The problem specifications will produce both $k_{eff}=1$ and the quoted $k_\infty$ to at least five decimal places. MCNP (Briesmeister, 1997) and DANTSYS (Alcouff, R.E, et al., 1995) have been verified using these problems. Additional uses of the test set for code verification are also discussed. Published by Elsevier Science Ltd.

Key words: analytic; benchmark; criticality; code verification

1 Introduction

This paper describes a set of benchmark problems with analytic eigenvalue ($k_{eff}$) and eigenfunction (flux) solutions to the neutron transport equation from peer-reviewed journal articles. The purpose of the test set is to verify that transport algorithms and codes can correctly calculate the analytic $k_{eff}$ and fluxes. The authors believe the reported eigenvalues and eigenfunctions to be accurate to at least five decimal places even though many references often report higher precision values. The higher precision eigenvalues and eigenfunctions from the references are reproduced here. These test set problems for infinite medium, slab, cylindrical, and spherical geometries in one- and

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two-energy groups, with one-, two-, and four-media, and isotropic and both linearly and quadratically anisotropic scattering are completely described using the listed references in this paper. A three-group infinite medium and a six-group variant $k_\infty$ problem (unpublished) are also included. This paper includes updates and minor corrections to previous Los Alamos papers and reports.\textsuperscript{13,14,15,16}

1.1 Verification and Validation

Verification is defined as "the process of evaluating a system or component to determine whether the products of a given development phase satisfy the conditions imposed at the start of the phase\textsuperscript{7}" or as a proof of correctness. Confirmation (proof) of correctness is defined as "a formal technique used to prove mathematically that a computer program satisfies its specified requirements.\textsuperscript{7}" In contrast to verification, validation is "the process of evaluating a system or component during or at the end of the development process to determine whether it satisfies specified requirements.\textsuperscript{7}" Thus code verification checks that the implemented code precisely reflects the intended calculations and that these calculations have been executed correctly. Code validation compares the accuracy of these calculated results usually with experimental data.

Figure 1 summarizes the process of modeling nature to developing a computer code simulation of radiation transport.\textsuperscript{8} One path to simulating radiation transport is to develop a theoretical physics model and describe it with math-
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Mathematical equations. Exact solutions to these complex theoretical equations are often impossible. Solutions to the mathematical equations require simplifying assumptions and are approximated with a carefully developed computer code. Code verification is the link between these mathematical and computational equations and the computer simulation. Verification of the code and data simulation includes comparisons of calculated results with analytic solutions and simplified verification data intended to be used only to verify computer code numerical performance. Other forms of code verification include comparisons with an accepted standard set of code output for regression testing, results from other computer codes, and line-by-line debugging.

A second path to characterizing radiation transport is to perform careful experiments and measure physical quantities using diagnostics. The accuracy of the measured diagnostics is limited by approximations and assumptions in the experimental methods and by the precision of the diagnostic equipment. Carefully designed experiments often infer or directly measure a desired physical parameter from the theoretical models and equations and thus are measurements of its true value. Code validation is the link between the measured diagnostics and the computer code with the general purpose physical data required by the computer simulation. Code and data validation includes comparison of the calculated code output with results from experiments and from other computer codes.

Figure 1 shows the process of developing a computer code simulation requires code verification, validation, and physical data. Importantly, the figure shows there is no direct path linking nature and computer simulation. Code verification must be performed before code validation. The objectives of this paper are to define and document a set of analytic eigenvalue and eigenfunction benchmarks for verifying criticality codes. Benchmark is defined as "a standard against which measurement or comparisons can be made."[7] Available benchmarks for code verification do not focus on criticality problems.[9] Validation benchmarks from critical experiments do exist, but are not verification benchmarks.[10] An initial effort to compile a benchmark test set for criticality calculation verification was begun, but not completed.[11],[12] The analytic benchmarks described here can be used to verify computed numerical solutions for $k_{eff}$ and the associated flux with virtually no uncertainty in the numerical benchmark values.

1.2 Why These Solutions Serve as a Test Set

All critical dimensions, $k_{eff}$, and scalar neutron flux results quoted here are based on numerical computations using the analytic solutions to the $k_{eff}$ eigenvalue (homogeneous) transport equation for "simple" problems. The analytic
methods used include Case's singular eigenfunction\textsuperscript{[13]}, \(F_N\) and \(S_N\) methods, \textsuperscript{[14],[15]} and Green's functions.\textsuperscript{[16]}

All of these test set problem specifications and results are from peer-reviewed journals, and have, in some cases, been solved numerically using more than one analytic solution. All calculated values for critical dimensions, \(k_{eff}\), and the scalar neutron flux are believed to be accurate to at least five decimal places. Several critical dimensions and scalar neutron flux are reported to more than five decimal places. The higher precision eigenvalues and eigenfunctions from the references are simply reproduced here.

1.3 Scope of the Criticality Verification Test Set

The verification test set was chosen to represent a "wide" range of problems from the relatively small number of published solutions. These problems include simple geometries, few neutron energy groups, and simplified (isotropic and linearly anisotropic) scattering models. The problems use neutron cross sections that are reasonable representations of the materials described. These cross sections are not general purpose multi-group values. The cross sections are used because they are extracted from the literature results and are intended to be used only to verify algorithm performance and not to predict criticality experiments.

The basic geometries include an infinite medium, slab, cylinder, and sphere with one- and two-energy group representations of uniform homogeneous materials. The slab and cylinder geometries are one-dimensional, as shown in Figure 2; that is, each is finite in one dimension (thickness for slab and radius for the cylinders) and infinite elsewhere. The two-media problems surround each geometry with a specified thickness of reflector. Solutions for one-, two-, and three-group infinite medium problems are derived in Appendix A.

The emphasis of the test set is on the fundamental eigenvalue, \(k_{eff}\). All \(k_{eff}\) eigenvalues for finite fissile materials are unity to at least five decimal places. The \(k_{\infty}\) values for a uniform homogeneous infinite medium are greater than unity. Few numerical eigenfunction solutions are published; consequently, mainly one-group and uniform homogeneous infinite medium fluxes are included in the test set results.

The critical dimension, \(r_c\), is defined pictorially for the one-dimensional, one-medium problem geometries in Figure 2, as well as the two-media infinite slab lattice cell. Reflector dimension(s) are provided for the reflected cases.
Fig. 2. Critical Dimension, $r_c$, for Bare One-Dimensional Geometries and Infinite Slab Lattice Cell

To assist in verification, each problem has a unique identifier. Since the test set includes bare and multi-media problems, there are two forms of the identifier. The first form is for a bare geometry:

**Fissile Material - Energy Groups - Scattering - Geometry**

The possible entries for each category are listed in Table 1. The fissile materials and identifier consist of Pu-239 (Pu), U-235 (U), highly enriched uranium-aluminum-water assembly (UAL), low enrichment uranium and D$_2$O reactor system (UD$_2$O), and a highly enriched uranium research reactor (URR). The identifier may be followed by a letter to differentiate between different cross section sets from nominally the same material. The table lists identifiers for the reflector material (if any), number of energy groups, scattering order, and geometry. The geometry is identified by the first two letters in the table. The exception is for the infinite slab lattice cell which uses ISLC. An example of the one material form of the identifier is:

**U-2-0-SP**

which is the identifier for a bare U-235 reactor (no reflector), 2 energy groups, isotropically scattering, in spherical geometry.

The second form of the identifier includes the reflecting material. The reflectors are usually H$_2$O with an exception of a three region Fe, Pb, Fe reflector. Although many of the reflectors are identified as H$_2$O, the reflector cross sections are unique to each problem. Consequently, a letter may follow H$_2$O indicating the H$_2$O cross section set used. The multi-media identifier form is:

**Fissile Material - Reflecting Material (thickness) - Energy Groups - Scattering - Geometry**

To separate multiple reflector thicknesses for the same fissile material, the
Table 1
Nomenclature for Problem Identifiers

<table>
<thead>
<tr>
<th>Fissile Material</th>
<th>Reflector Material</th>
<th>Energy Groups</th>
<th>Scattering Order</th>
<th>Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>PU</td>
<td>bare</td>
<td>1 group</td>
<td>0 - $P_0$ Isotropic</td>
<td>Infinite</td>
</tr>
<tr>
<td>U</td>
<td>H$_2$O</td>
<td>2 groups</td>
<td>1 - $P_1$ Anisotropic</td>
<td>Slab</td>
</tr>
<tr>
<td>UD$_2$O</td>
<td>Fe-Na</td>
<td>3 groups</td>
<td>2 - $P_2$ Anisotropic</td>
<td>Cylinder</td>
</tr>
<tr>
<td>UAL</td>
<td></td>
<td>6 groups</td>
<td></td>
<td>Sphere</td>
</tr>
<tr>
<td>URR</td>
<td></td>
<td></td>
<td></td>
<td>Infinite Slab Lattice Cell</td>
</tr>
</tbody>
</table>

Thickness is given in parenthesis in the title in units of mean free paths (mfp). For example,

UD$_2$O-H$_2$O(10)-1-0-SL

is the identifier for a uranium and D$_2$O reactor with a H$_2$O reflector of 10 mean free path thickness, one-energy group, isotropically scattering, in slab geometry. An ‘IN’ in parenthesis after the H$_2$O means an infinite water reflector.

There are 43 problems in the one-energy group case; 30 problems assume isotropic scattering and 13 have anisotropic scattering. For the two-energy group problems, there are 30 problems subdivided into 26 isotropic scattering problems and 4 linearly anisotropic problems. Also included for an infinite medium are a three-group and a six-group (2 coupled sets of three groups) isotropic problem. The test set includes 24 infinite medium problems, 24 slabs, 9 one-energy group cylinders, 14 spheres, and 4 infinite slab lattice cells.

2 Uses of the Criticality Verification Test Set

This paper provides all necessary problem definitions and published critical dimensions, $k_{eff}$, and scalar neutron flux results to verify a criticality transport algorithm or code and associated numerics such as random number generation and round-off errors. All material cross sections provided are macroscopic, so the atom density used by the code should be unity. Cross section values are assumed accurate to the number of decimal places reported. Not all of the analytic solutions from the references are used, however, because the number of problems in the test set becomes too large. For other solutions not included in this paper, see the reference list.

The verification test set problems can be used in several ways. The user can choose to simply calculate the problems and compare forward and adjoint
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$k_{eff}$ and neutron flux results with the benchmark solutions. However, there are several more verification processes that could be included. For example, in Monte Carlo codes, examining two forms of cross sections representation we might include multi-group and pointwise representation of multi-group data. In multi-group problems, an alternative verification procedure is to change the energy group structure when up-scattering is allowed; that is, reverse the order of the fast and slow groups.

Another part of code verification is testing different representations of the same geometry (e.g. reflecting boundaries and lattices). An example is an infinite one-dimensional slab (finite in one dimension and infinite in the other two dimensions) as shown in Figure 2, which could be modeled as a three-dimensional cube with four reflective boundaries. Other geometry options can be tested by constructing several smaller cubes inside of the three-dimensional representation of a one-dimensional critical slab. The infinite medium problem can be represented by using large geometric boundaries, reflecting boundaries, or infinite lattices of finite shapes. Infinite medium problems can be used to verify constant scalar and angular flux in each energy group as well as scalar flux ratios for more than one energy group. Three-dimensional geometric representations of optically small objects can also be tested for $k_\infty$ in infinite medium problems.[17] Purely absorbing one-group infinite medium problems can provide faster code verification since scattering does not alter the infinite medium $k_\infty$ (see Appendix A).

Another use of this verification test set includes testing of any flux approximations. This can be especially important at near tangential angles where many codes assume a value for the incident angle. This can also affect $k_{eff}$ if it is estimated by same section of code that calculates the flux.

Different calculation capabilities of a code should be tested using these problems. For Monte Carlo codes, different variance reduction methods such as analog or implicit capture and geometric splitting or Russian roulette can be verified. Cycle-to-cycle correlations in the estimated $k_{eff}$ standard deviation must be taken into account to form valid $k_{eff}$ confidence intervals. Statistically independent runs can be made and analyzed if necessary. The magnitude of any negative bias in $k_{eff}$, which is a function of the number of neutron histories per fission generation, also needs to be considered and made smaller than 0.00001. [18]

Deterministic codes can assess convergence characteristics and correctness of $k_{eff}$ and the flux as a function of space and angle representation. Various characteristics of discrete ordinates numerics can also be checked such as the effects of eigenvalue search algorithms, angular redistribution terms in curvilinear geometries, ray effects, and various alternative geometric descriptions.
3 Neutron Transport Equation Overview

The neutron transport equation being solved in these benchmark problems is briefly described for one- and two-energy groups and the isotropic and linearly anisotropic cases. The infinite medium solutions for $k_{\infty}$ and the flux ratios are described in Appendix A.

3.1 General $k_{\text{eff}}$ Eigenvalue Equation

The steady state neutron transport equation can be written as a $k_{\text{eff}}$ eigenvalue problem as: \[ \bar{\Omega} \cdot \nabla \Psi(\vec{r}, E, \vec{\Omega}) + \Sigma_t(\vec{r}, E) \Psi(\vec{r}, E, \vec{\Omega}) = \int dE' \int \frac{d\Omega'}{4\pi} \Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega}) \Psi(\vec{r}, E', \vec{\Omega}') + \chi(E) \int \frac{dE'}{4\pi k_{\text{eff}}} \Sigma_f(\vec{r}, E') \int \frac{\Psi(\vec{r}, E', \vec{\Omega}') d\Omega'}{4\pi} \] (1)

where:

$\Psi(\vec{r}, E, \vec{\Omega})$ = angular neutron flux as a function of space $\vec{r}$, energy $E$, and angle $\vec{\Omega}$

$\Sigma_t(\vec{r}, E)$ = total neutron macroscopic cross section

$\Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega}) dE d\Omega$ = neutron scattering macroscopic cross section from $E'$ to $E+dE$ in direction $d\Omega'$ about $\vec{\Omega}$

$\Sigma_f(\vec{r}, E')$ = neutron fission macroscopic cross section

$\nu(\vec{r}, E')$ = number of neutrons emitted from each fission event

$\chi(E)$ = fission neutron energy distribution

For these test problems, there are no $(n, x\alpha')$ reactions, $x > 1$, included in $\Sigma_s$. Therefore, $\Sigma_c = \Sigma_t - \Sigma_s - \Sigma_f$, where $\Sigma_c$ is the neutron capture cross section (zero neutrons emitted). This paper also provides values for the scalar neutron flux, which is defined as $\phi(\vec{r}, E) = \int_{\Omega} \Psi(\vec{r}, E, \vec{\Omega}) d\Omega$. The reported scalar neutron flux values are normalized to the flux at the center of the fissile material.
The $k_{eff}$ eigenvalue is only associated with the fission reaction and no other multiplying process such as $(n, 2n)$. The fundamental eigenvalue, $k_{eff}$, is unity for a critical system, less than unity for a subcritical system, and greater than unity for a supercritical system. The steady state $k_{eff}$ eigenvalue equation is physically correct only when $k_{eff}$ is unity and there is no decay or growth in $\Psi(r, E, \Omega)$. A solution when $k_{eff}$ is not unity is still a valuable indicator of the ability of a system to sustain a fission chain reaction. When an infinite medium is considered, $k_{eff}$ will be referred to as $k_\infty$. This paper gives results for the fundamental eigenvalue. For higher eigenvalue results see references [20], [21], [22], [24], [25], [26], [27].

3.2 One-Energy Group in One-Dimensional Slab Geometry

3.2.1 Isotropic Scattering.

The one-energy group, homogeneous medium with isotropic scattering transport equation for a slab can be written in terms of an optical thickness ($z = \Sigma_t x$) as: \cite{19}

$$\mu \frac{\partial \Psi(z, \mu)}{\partial z} + \Psi(z, \mu) = \frac{c}{2} \int_{-1}^{1} \Psi(z, \mu') d\mu'$$

where:

$$c = \frac{\Sigma_s + \frac{\nu \Sigma_f}{k_{eff}}}{\Sigma_t}$$

The eigenvalue form of $c$ is defined as the mean number of secondary neutrons produced per neutron reaction and is also known as the secondaries ratio. This equation still requires elaborate mathematics to solve as reported in the literature. Derivation of the one-energy group $k_\infty$ solution for the infinite medium case is shown in Appendix A.

The literature uses this form of the neutron transport equation with a non-reentrant boundary condition to derive one-energy group, isotropic scattering analytic solutions for the critical ($k_{eff} = 1$) dimensional scalar neutron flux. It should be noted that $c$ values for $k_{eff} = 1$ in equation 3 are presented in the literature, thereby making $c = \frac{\Sigma_s + \nu \Sigma_f}{\Sigma_t}$. The typical range of $c$ found in the literature for fissile materials is from 1.01 to 2.00. The one-group cross sections selected for the test set mimic the physical characteristics of the two-group problems and range from 1.02 to 1.50. A value of $c$ of 1.5 is the upper limit for real fissile materials.
3.2.2 Linearly Anisotropic Scattering.

The scattering term, \( \Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega}) \) can be a strong function of the cosine of the scattering angle, \( \mu_0 = \vec{\Omega}' \cdot \vec{\Omega} \). The angular dependence can be analyzed by Legendre polynomial series expansion of \( \Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \cdot \vec{\Omega}) \). Using the Legendre polynomial expansion, the one-energy group, one-dimensional slab, neutron transport equation can be written in a similar form to equation 2:\([28]\)

\[
\mu \frac{\partial \Psi(z, \mu)}{\partial z} + \Psi(z, \mu) = \frac{c}{2} \int_{-1}^{1} \Psi(z, \mu')(1 + \mu' \mu) d\mu'
\]

This form of the neutron transport equation is often found in the literature. The solution of this equation includes linearly anisotropic scattering; however, it also includes a linearly anisotropic fission source emission. For problems that include the anisotropic effect on the fission term, see references [22], [29], [30]. Numerical solutions exist that do not force the anisotropic effect on the fission term. This limitation on the different anisotropic behavior of scattering and fission can be removed by using different transfer functions for scattering and fission. Solutions for the higher eigenvalues exist for this form of the transport equation.\([29],[21],[22],[24],[25],[26],[27]\)

3.3 Two-Energy Groups in One-Dimensional Slab Geometry

3.3.1 Isotropic Scattering.

Using the same procedures as in the one-group case, the two-energy group form of the transport equation for a slab can be written as: \([15]\)

\[
\mu \frac{\partial \Psi_1(x, \mu)}{\partial x} + \Sigma_1 \Psi_1(x, \mu) = \frac{\Sigma_{11}}{2} \int_{-1}^{1} \Psi_1(x, \mu') d\mu' + \frac{\Sigma_{12}}{2} \int_{-1}^{1} \Psi_2(x, \mu') d\mu'
\]

\[
\mu \frac{\partial \Psi_2(x, \mu)}{\partial x} + \Sigma_2 \Psi_2(x, \mu) = \frac{\Sigma_{21}}{2} \int_{-1}^{1} \Psi_1(x, \mu') d\mu' + \frac{\Sigma_{22}}{2} \int_{-1}^{1} \Psi_2(x, \mu') d\mu'
\]

where:

\( \Sigma_i = \text{total neutron macroscopic cross section of group } i \)

\( \Sigma_{ij} = \text{total neutron group transfer macroscopic cross section from group } j \text{ to group } i \)
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In this paper, the fast energy group is group 2 to be consistent with most of the references. This notation is the reverse of most nuclear engineering textbooks.

Assuming group 2 is the fast group and no non-fission up-scatter for the slow group, the group transfer cross sections are given by:

\[
\begin{align*}
\Sigma_{11} &= \Sigma_{1f} + \chi_1 \nu_1 \Sigma_{1f}/k_{\text{eff}} \\
\Sigma_{21} &= \chi_2 \nu_1 \Sigma_{1f}/k_{\text{eff}} \\
\Sigma_{12} &= \Sigma_{1s} + \chi_1 \nu_2 \Sigma_{2f}/k_{\text{eff}} \\
\Sigma_{22} &= \Sigma_{2s} + \chi_2 \nu_2 \Sigma_{2f}/k_{\text{eff}}
\end{align*}
\]

Note that \( \Sigma_i = \Sigma_{ic} + \Sigma_{if} + \Sigma_{its} + \Sigma_{jis} \), where the \( \Sigma_{jis} \) represents nonfission scattering to group \( j \neq i \). This equation for \( \Sigma_i \) again assumes that the \( \Sigma(n,2n)_i \) components are zero.

The two-energy group form of the transport equation, which has solutions in the literature, can be written in a similar form to the one-group equations utilizing the optical thickness parameter, \( z = \Sigma_2 x \), but in matrix-vector notation as seen below.

\[
\mu \frac{\partial \vec{\Psi}(z, \mu)}{\partial z} + \vec{\Sigma} \vec{\Psi}(z, \mu) = \frac{\gamma}{2} \int_{-1}^{1} \vec{\Psi}(z, \mu') d\mu'
\]

where:

\[
\vec{\Psi}(z, \mu) = \begin{bmatrix} \Psi_1(z, \mu) \\ \Psi_2(z, \mu) \end{bmatrix}, \quad \vec{\Sigma} = \begin{bmatrix} \Sigma & 0 \\ 0 & 1 \end{bmatrix}, \quad \vec{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}
\]

and

\[
c_{ij} = \Sigma_{ij}/\Sigma_2.
\]

The derivations for the infinite medium \( k_\infty \) and the group 2 to group 1 flux ratio are given in Appendix A.
3.3.2 Linearly Anisotropic Scattering.

One of the above simplifying assumptions to the steady state neutron transport equation is that neutron scattering is isotropic (no angular dependence). However, the scattering term, $\Sigma_s(\vec{r}', E' \rightarrow E, \Omega' \rightarrow \Omega)$ can be a strong function of the cosine of the scattering angle, $\mu_0 = \Omega' \cdot \Omega$. This angular dependence can be analyzed by Legendre polynomial series expansion of $\Sigma_s(\vec{r}', E' \rightarrow E, \Omega' \cdot \Omega)$ giving

$$\Sigma_s(\vec{r}', E' \rightarrow E, \Omega' \cdot \Omega) = \sum_{l=0}^{M} \frac{2l+1}{4\pi} \Sigma_{sl}(\vec{r}', E' \rightarrow E)P_l(\Omega' \cdot \Omega)$$

where $M$ indicates the degree of anisotropy. For $M = 0$, scattering in the lab system is isotropic and for $M = 1$, scattering is linearly anisotropic. A complete mathematical description is in reference [19] and [28]. For linearly anisotropic scattering, the scattering cross section for general anisotropic scattering consists of two components, $\Sigma_{s0}$ and $\Sigma_{s1}$, where $\Sigma_{s1}$ is the linear anisotropic scattering component and affects the scattering angular distribution for both in and out of group scattering. Anisotropic scattering can be forward or backward peaked and thus $\Sigma_{s1}$ can be positive or negative. The total scattering cross section is not dependent on $\Sigma_{s1}$. The anisotropic cross section only affects the angular distribution. Infinite medium $k_\infty$ and neutron flux results are independent of the anisotropic cross section.

Following the same procedures, the general two-speed linearly anisotropically scattering analogue to equation 6 which also has numerical solutions is:[32]

$$\mu \frac{\partial \tilde{\Psi}(z, \mu)}{\partial z} + \tilde{\Sigma} \tilde{\Psi}(z, \mu) = \frac{1}{2} \sum_{l=0}^{1} \tilde{C}_l P_l(\mu) \int_{-1}^{1} \tilde{\Psi}(z, \mu') P_l(\mu') d\mu'$$

where:

$$\tilde{\Psi}(z, \mu) = \begin{bmatrix} \Psi_1(z, \mu) \\ \Psi_2(z, \mu) \end{bmatrix}, \tilde{\Sigma} = \begin{bmatrix} \Sigma & 0 \\ 0 & 1 \end{bmatrix}, \tilde{C}_l = \begin{bmatrix} c_{11l} & c_{12l} \\ c_{21l} & c_{22l} \end{bmatrix}$$

and

$$c_{ijl} = (2l + 1)\Sigma_{ijl}/\Sigma_2.$$

The $\Sigma_{ijs}$ term is the linearly anisotropic scattering cross section and is given
in the problem descriptions without the \((2l + 1)\) term.

4 One-Energy Group Problem Definitions and Results

For the one-energy group cases, the critical dimension(s) for each geometry depends upon the \(c\) value chosen from the literature and not specific cross section sets. To use the literature results, the cross sections were selected to match published \(c\) values with \(k_{\text{eff}} = 1\) at low, middle, and high \(c\) values listed. Values ranging from 1.02, 1.30, 1.40, and 1.50 were chosen because they are similar to the physical systems in the two-group cases: uranium-D\(_2\)O reactor, U-235, and Pu-239. These problems use cross sections that are reasonable representations of these materials; however, these cross sections are not general purpose one-group values. The cross sections are used because they define the \(c\) values used in the literature and are intended to be used only to verify algorithm performance and not to predict any actual criticality experiments. Cross section values are assumed accurate to the number of decimal places reported.

The isotropic neutron macroscopic cross sections provided for each case are: the total cross section, \(\Sigma_t\), the capture (no neutrons emitted) cross section, \(\Sigma_c\), the scattering cross section, \(\Sigma_s\), the fission cross section, \(\Sigma_f\), and the number of neutrons, \(\nu\), emitted for each fission. The \((n,2n), (n,3n), \ldots\) cross sections are assumed to be zero (but need not be). Thus the total cross section equals the sum of \(\Sigma_c, \Sigma_s,\) and \(\Sigma_f\), thereby providing a consistency check on the cross section set. Many references give \((\nu \Sigma_f)\) instead of \(\nu\) and \(\Sigma_f\). Since both parameters (not the product) may be required by a code for the problem solution, the product \((\nu \Sigma_f)\) has been split into \(\nu\) and \(\Sigma_f\) preserving their product and \(\Sigma_t\). The value of \(c\) for \(k_{\text{eff}} = 1\) in equation 6 is also included in each cross section table. For the reflected spheres, different secondaries ratios, \(c\), are reported with the critical dimension for \(k_{\text{eff}} = 1\) for various combinations of core and reflector thicknesses. To maintain consistent cross sections with the U-235 set, the parameter, \(\nu\), was modified to match \(c\) to the literature values.

When anisotropic scattering cross sections are provided, the anisotropic components are designated by \(\Sigma_{s1}\) and \(\Sigma_{s2}\), respectively. Similarly, the isotropic scattering component is designated by \(\Sigma_{s0}\).

The value of \(k_{\infty}\), as defined in Appendix A, is given for each cross section set. For finite problems where \(k_{\text{eff}}\) is unity, the critical dimension, \(r_c\), is listed for each geometry in both mean free paths (to indicate the neutron optical thickness) and in centimeters for the one-dimensional geometries. When available in the literature, the scalar flux values, normalized to the flux at the center
of the fissile material, are also provided. The two-media problems have $c_1 > 1$ for the core region 1 and $c_2 < 1$ for the surrounding reflector region 2. The two-media problems use the cross sections for the nonmultiplying reflector. The critical dimensions for the multiplying medium and reflector thickness are given in both mean free paths and centimeters.

A comparison of the critical dimensions for the different geometries behave as expected; that is, the critical dimension is smallest for the one-dimensional slab and increases for the cylinder and sphere. This behavior is to be expected due to the increased leakage with the curvi-linear geometries. For the reflected geometries, the critical dimension decreases with increasing reflector thickness.
4.1 One-Energy Group Isotropic Scattering

4.1.1 One-Group Pu-239.

One-Energy Group Isotropic Cross Sections

Table 2 gives the one-group, isotropic cross sections for two cases of Pu-239 ($c=1.50$ and $c=1.40$) and a H$_2$O ($c=0.90$) reflector. The total cross sections are the same for both Pu-239 cases and H$_2$O as required by the reference for the two-media solutions.

Table 2
One-Group Macroscopic Cross Sections ($cm^{-1}$) for Pu-239 ($c=1.40,1.50$) and H$_2$O ($c=0.90$)

<table>
<thead>
<tr>
<th>Material</th>
<th>$\nu$</th>
<th>$\Sigma_f$</th>
<th>$\Sigma_c$</th>
<th>$\Sigma_s$</th>
<th>$\Sigma_t$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pu-239 (a)</td>
<td>3.24</td>
<td>0.081600</td>
<td>0.019584</td>
<td>0.225216</td>
<td>0.32640</td>
<td>1.50</td>
</tr>
<tr>
<td>Pu-239 (b)</td>
<td>2.84</td>
<td>0.081600</td>
<td>0.019584</td>
<td>0.225216</td>
<td>0.32640</td>
<td>1.40</td>
</tr>
<tr>
<td>H$_2$O (refl)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.032640</td>
<td>0.293760</td>
<td>0.32640</td>
<td>0.90</td>
</tr>
</tbody>
</table>

Infinite Medium (PUa-1-0-IN and PUB-1-0-IN)

Using the cross sections for Pu-239 (a) (problem 1) in Table 2, $k_{\infty} = 2.612903$ with a constant angular and scalar flux everywhere. Using the cross sections for Pu-239 (b) (problem 5) in Table 2, $k_{\infty} = 2.290323$ with a constant angular and scalar flux everywhere.

One-Medium Slab, Cylinder, and Sphere Critical Dimensions

The Pu-239 (a) critical dimension, $r_c$, is listed in Table 3.

Table 3
Critical Dimensions, $r_c$, for One-Group Bare Pu-239 ($c=1.50$)

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>$r_c$ (mfp)</th>
<th>$r_c$ (cm)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>PUa-1-0-SL</td>
<td>Slab</td>
<td>0.605055</td>
<td>1.853722</td>
<td>[16]</td>
</tr>
</tbody>
</table>

The Pu-239 (b) critical dimensions, $r_c$, are listed in Table 4. The normalized scalar flux for four spatial positions are given in Table 5 using the same references. The flux ratios for PUB-1-0-CY are only available to four decimal places.
Table 4
Critical Dimensions, \( r_c \), for One-Group Bare Pu-239 (\( c = 1.40 \))

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>( r_c ) (mfp)</th>
<th>( r_c ) (cm)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>PUb-1-0-SL</td>
<td>Slab</td>
<td>0.73660355</td>
<td>2.256751</td>
<td>[35]</td>
</tr>
<tr>
<td>7</td>
<td>PUb-1-0-CY</td>
<td>Cylinder</td>
<td>1.396979</td>
<td>4.279960</td>
<td>[36],[37]</td>
</tr>
<tr>
<td>8</td>
<td>PUb-1-0-SP</td>
<td>Sphere</td>
<td>1.9853434324</td>
<td>6.082547</td>
<td>[35]</td>
</tr>
</tbody>
</table>

Table 5
Normalized Scalar Fluxes for One-Group Bare Pu-239 (\( c = 1.40 \))

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>( r/r_c = 0.25 )</th>
<th>( r/r_c = 0.5 )</th>
<th>( r/r_c = 0.75 )</th>
<th>( r/r_c = 1.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>PUb-1-0-SL</td>
<td>Slab</td>
<td>0.9701734</td>
<td>0.8810540</td>
<td>0.7318131</td>
<td>0.4902592</td>
</tr>
<tr>
<td>7</td>
<td>PUb-1-0-CY</td>
<td>Cylinder</td>
<td>—</td>
<td>0.8093</td>
<td>—</td>
<td>0.2926</td>
</tr>
<tr>
<td>8</td>
<td>PUb-1-0-SP</td>
<td>Sphere</td>
<td>0.93538006</td>
<td>0.75575352</td>
<td>0.49884364</td>
<td>0.19222603</td>
</tr>
</tbody>
</table>

Two-Media Slab and Cylinder Critical Dimensions

The literature values in Tables 6 and 7 give the critical dimensions for Pu-239 (a) for two H\(_2\)O reflector thicknesses. The first two-media problem (problem 3) in Table 6 is a special nonsymmetric two-region, Pu-239 and H\(_2\)O, problem. The second two-media problem (problem 4) in Table 7 is a symmetric three-region problem with the reflector on both sides of the fissile medium.

Table 6
Critical Dimensions for One-Group Pu-239 Slab (\( c = 1.50 \)) with Non-Symmetric H\(_2\)O Reflector (\( c = 0.90 \))

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>Pu ( r_c )</th>
<th>H(_2)O thickness</th>
<th>Pu+H(_2)O Radius</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>PUa-H(_2)O(1.1-0.6)-Slab</td>
<td>Slab (mfp) (cm)</td>
<td>0.482566</td>
<td>1</td>
<td>1.478450</td>
<td>4.542175</td>
</tr>
</tbody>
</table>

Table 7
Critical Dimensions for One-Group Pu-239 Slab (\( c = 1.50 \)) with H\(_2\)O Reflector (\( c = 0.90 \))

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>Pu ( r_c )</th>
<th>H(_2)O thickness</th>
<th>Pu+H(_2)O radius</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>PUa-H(_2)O(0.5)-1-0-SL</td>
<td>Slab (mfp) (cm)</td>
<td>0.43015</td>
<td>0.5</td>
<td>1.317862</td>
<td>1.531863</td>
</tr>
</tbody>
</table>

The literature values in Table 8 give the critical dimensions for Pu-239 (b) with two H\(_2\)O reflector thicknesses.
Table 8
Critical Dimensions for One-Group Pu-239 Cylinder (c=1.40) with H₂O Reflector (c=0.90)

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>Pu rₐ</th>
<th>H₂O thickness</th>
<th>Pu+H₂O Radius</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>PUb-H₂O(1)-1-0-CY</td>
<td>Cylinder (mfp)</td>
<td>1.10898</td>
<td>1</td>
<td>6.401335</td>
<td>[38]</td>
</tr>
<tr>
<td>10</td>
<td>PUb-H₂O(10)-1-0-CY</td>
<td>Cylinder (mfp)</td>
<td>1.00452</td>
<td>10</td>
<td>33.714829</td>
<td>[38]</td>
</tr>
</tbody>
</table>

4.1.2 One-Group U-235.

One-Group Isotropic Cross Sections

Table 9 gives the one-group, isotropic cross sections for two cases of U-235 and a H₂O reflector. Notice that one-group Σᵣ for Pu-239 and U-235 are the same as given in reference [41], but the secondaries ratio, c, differs.

Table 9
One-Group Macroscopic Cross Sections (cm⁻¹) for U-235 (c=1.30)

<table>
<thead>
<tr>
<th>Material</th>
<th>ν</th>
<th>Σᵣ</th>
<th>Σₛ</th>
<th>Σᵣ</th>
<th>Σₛ</th>
<th>Σₜ</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>U-235 (a)</td>
<td>2.70</td>
<td>0.065280</td>
<td>0.013056</td>
<td>0.248064</td>
<td>0.32640</td>
<td>1.30</td>
<td></td>
</tr>
<tr>
<td>U-235 (b)</td>
<td>2.797101</td>
<td>0.065280</td>
<td>0.013056</td>
<td>0.248064</td>
<td>0.32640</td>
<td>1.3194202</td>
<td></td>
</tr>
<tr>
<td>U-235 (c)</td>
<td>2.707308</td>
<td>0.065280</td>
<td>0.013056</td>
<td>0.248064</td>
<td>0.32640</td>
<td>1.3014616</td>
<td></td>
</tr>
<tr>
<td>U-235 (d)</td>
<td>2.679198</td>
<td>0.065280</td>
<td>0.013056</td>
<td>0.248064</td>
<td>0.32640</td>
<td>1.3014616</td>
<td></td>
</tr>
<tr>
<td>H₂O (refl)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.032640</td>
<td>0.293760</td>
<td>0.32640</td>
<td>0.32640</td>
<td>0.90</td>
</tr>
</tbody>
</table>

Infinite Medium (UA-1-0-IN, UB-1-0-IN, UC-1-0-IN, and UD-1-0-IN)

Using the cross sections for U-235 (a) in Table 9, kᵪ = 2.250000 (problem 11) with a constant angular and scalar flux everywhere. Using the cross sections for U-235 (b), U-235 (c), and U-235 (d) in Table 9, kᵪ = 2.330917 (problem 15), 2.256083 (problem 17), and 2.232667 (problem 19) with a constant angular and scalar flux everywhere, respectively.

One-Medium Slab, Cylinder, and Sphere Critical Dimensions

The critical dimension, rₑ, and spatial flux ratios are given in Table 10 and 11 for U-235 (a). The references are the same for both tables.
Table 10
Critical Dimensions, \( r_c \), for One-Group Bare U-235 (\( c=1.30 \))

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>( r_c ) (mfp)</th>
<th>( r_c ) (cm)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>Ua-1-0-SL</td>
<td>Slab</td>
<td>0.93772556</td>
<td>2.872934</td>
<td>[35]</td>
</tr>
<tr>
<td>13</td>
<td>Ua-1-0-CY</td>
<td>Cylinder</td>
<td>1.72500292</td>
<td>5.284935</td>
<td>[36], [37]</td>
</tr>
<tr>
<td>14</td>
<td>Ua-1-0-SP</td>
<td>Sphere</td>
<td>2.4248249802</td>
<td>7.428998</td>
<td>[35]</td>
</tr>
</tbody>
</table>

Table 11
Normalized Scalar Fluxes for One-Group Bare U-235 (\( c=1.30 \))

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>( r/r_c = 0.25 )</th>
<th>( r/r_c = 0.5 )</th>
<th>( r/r_c = 0.75 )</th>
<th>( r/r_c = 1.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>Ua-1-0-SL</td>
<td>Slab</td>
<td>0.9669506</td>
<td>0.8686259</td>
<td>0.7055218</td>
<td>0.4461912</td>
</tr>
<tr>
<td>14</td>
<td>Ua-1-0-SP</td>
<td>Sphere</td>
<td>0.93244907</td>
<td>0.74553332</td>
<td>0.48095413</td>
<td>0.17177706</td>
</tr>
</tbody>
</table>

Two-Media Sphere Critical Dimensions

The literature values in Table 12, give the critical dimensions for U-235 (b), U-235 (c), and U-235 (d) for three spherical \( H_2O \) reflector thicknesses.

Table 12
Critical Dimensions for One-Group U-235 Sphere with \( H_2O \) Reflector (\( c=0.90 \))

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>( U )</th>
<th>( r_c )</th>
<th>( H_2O ) thickness</th>
<th>( U+H_2O ) Radius</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>Ub-H2O(1)-1-0-SP</td>
<td>Sphere (mfp)</td>
<td>2</td>
<td>1</td>
<td>3.063725</td>
<td>9.191176</td>
<td>[24],[27]</td>
</tr>
<tr>
<td>18</td>
<td>Uc-H2O(2)-1-0-SP</td>
<td>Sphere (mfp)</td>
<td>2</td>
<td>2</td>
<td>6.12745</td>
<td>12.2549</td>
<td>[24],[27]</td>
</tr>
<tr>
<td>20</td>
<td>Ud-H2O(3)-1-0-SP</td>
<td>Sphere (mfp)</td>
<td>2</td>
<td>3</td>
<td>6.12745</td>
<td>15.318626</td>
<td>[24],[27]</td>
</tr>
</tbody>
</table>

4.1.3 One-Group \( U-D_2O \) Reactor.

One-Group Isotropic Cross Sections

Table 13 gives the one-group, isotropic cross sections for the uranium-\( D_2O \) reactor and \( H_2O \) reflector. Note that the uranium-\( D_2O \) reactor and \( H_2O \) reflector have the same total cross section as required by the references for the reflected cylindrical solutions.
Table 13
One-Group Macroscopic Cross Sections (cm\(^{-1}\)) for U-D\(_2\)O Reactor (\(c=1.02\)) and H\(_2\)O (\(c=0.90\))

<table>
<thead>
<tr>
<th>Material</th>
<th>(\nu)</th>
<th>(\Sigma_f)</th>
<th>(\Sigma_c)</th>
<th>(\Sigma_s)</th>
<th>(\Sigma_t)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>U-D(_2)O</td>
<td>1.70</td>
<td>0.054628</td>
<td>0.027314</td>
<td>0.464338</td>
<td>0.54628</td>
<td>1.02</td>
</tr>
<tr>
<td>H(_2)O (refl)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.054628</td>
<td>0.491652</td>
<td>0.54628</td>
<td>0.90</td>
</tr>
</tbody>
</table>

Infinite Medium (UD2O-1-0-IN)

Using the cross sections for U-D\(_2\)O in Table 13, \(k_\infty = 1.133333\) (problem 21) with a constant angular and scalar flux everywhere.

One-Medium Slab, Cylinder, and Sphere Critical Dimensions

The critical dimension, \(r_c\), and spatial flux ratios are listed in Table 14 and 15.

Table 14
Critical Dimensions, \(r_c\), for One-Group Bare U-D\(_2\)O Reactor (\(c=1.02\))

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>(r_c) (mfp)</th>
<th>(r_c) (cm)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>22</td>
<td>UD2O-1-0-SL</td>
<td>Slab</td>
<td>5.6655054562</td>
<td>10.371065</td>
<td>[35]</td>
</tr>
<tr>
<td>23</td>
<td>UD2O-1-0-CY</td>
<td>Cylinder</td>
<td>9.043255</td>
<td>16.554249</td>
<td>[36],[37]</td>
</tr>
<tr>
<td>24</td>
<td>UD2O-1-0-SP</td>
<td>Sphere</td>
<td>12.0275320980</td>
<td>22.017156</td>
<td>[35]</td>
</tr>
</tbody>
</table>

Table 15
Normalized Scalar Fluxes for One-Group Bare U-D\(_2\)O Reactor (\(c=1.02\))

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>(r/r_c = 0.25)</th>
<th>(r/r_c = 0.5)</th>
<th>(r/r_c = 0.75)</th>
<th>(r/r_c = 1.0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>22</td>
<td>UD2O-1-0-SL</td>
<td>Slab</td>
<td>0.93945236</td>
<td>0.76504084</td>
<td>0.49690627</td>
<td>0.13893858</td>
</tr>
<tr>
<td>24</td>
<td>UD2O-1-0-SP</td>
<td>Sphere</td>
<td>0.91063756</td>
<td>0.67099621</td>
<td>0.35561622</td>
<td>0.04678614</td>
</tr>
</tbody>
</table>

Two-Media Slabs and Cylinders Critical Dimensions

Table 16 gives the U-D\(_2\)O critical dimension, \(r_c\), for two H\(_2\)O reflector thicknesses.
Table 16
Critical Dimensions for One-Group U-D_2O (c=1.02) Slab and Cylinder with H_2O (c=0.90) Reflector

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>UD_2O ( r_c )</th>
<th>H_2O thickness</th>
<th>UD_2O + H_2O radius</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>UD2O-H20(1)-1-0-SL</td>
<td>Slab (mfp)</td>
<td>5.0335</td>
<td>1</td>
<td>11.044702</td>
<td>[42], [43]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(cm)</td>
<td>9.214139</td>
<td>1.830563</td>
<td></td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>UD2O-H20(10)-1-0-SL</td>
<td>Slab (mfp)</td>
<td>4.6041</td>
<td>10</td>
<td>26.733726</td>
<td>[42],[43]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(cm)</td>
<td>8.428096</td>
<td>18.30563</td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>UD2O-H20(1)-1-0-CY</td>
<td>Cylinder (mfp)</td>
<td>15.396916</td>
<td>1.830563</td>
<td>17.227479</td>
<td>[38]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(cm)</td>
<td>7.979325</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>UD2O-H20(10)-1-0-CY</td>
<td>Cylinder (mfp)</td>
<td>14.606658</td>
<td>18.30563</td>
<td>32.912288</td>
<td>[38]</td>
</tr>
</tbody>
</table>

4.1.4 One-Group U-235 Reactor

One-Group Isotronic Cross Sections

Table 17 gives the one-group, isotropic cross sections for the U-235 reactor with a Fe reflector and Na moderator.

Table 17
One-Group Macroscopic Cross Sections (cm\(^{-1}\)) for U-235 Reactor, Fe reflector, and Na Moderator

<table>
<thead>
<tr>
<th>Material</th>
<th>( \nu )</th>
<th>( \Sigma_f )</th>
<th>( \Sigma_c )</th>
<th>( \Sigma_s )</th>
<th>( \Sigma_t )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>U-235 (e)</td>
<td>2.50</td>
<td>0.06922744</td>
<td>0.01013756</td>
<td>0.328042</td>
<td>0.407407</td>
<td>1.230</td>
</tr>
<tr>
<td>Fe (refl)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.00046512</td>
<td>0.23209488</td>
<td>0.23256</td>
<td>0.9980</td>
</tr>
<tr>
<td>Na (mod)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.086368032</td>
<td>0.086368032</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Infinite Medium (Ue-1-0-IN)

Using the cross sections for the U-235 reactor in Table 17, \( \kappa_\infty = 2.1806667 \) (problem 29) with a constant angular and scalar flux everywhere.

One-Medium Slab Critical Dimensions

Note that this problem is a nonsymmetric four-region problem. The U-235 is surrounded by a Fe cladding on two sides but moderated by Na on one side. The critical dimension, \( r_c \), is listed in Tables 18 and 19.
The U-235 (e) critical dimensions, $r_c$, are listed in Tables 18 and 19. The normalized scalar flux for four spatial positions are given in Table 20 using the same references. These positions correspond to the material boundaries and are normalized by the scalar neutron flux at the left boundary.

Table 20
Normalized Scalar Fluxes for One-Group U-235 Reactor

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>Fe-U</th>
<th>U-Fe</th>
<th>Fe-Na</th>
<th>Na</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>Ue-Fe-Na-1-0-SL</td>
<td>Slab</td>
<td>1.229538</td>
<td>1.49712</td>
<td>1.324899</td>
<td>0.912273</td>
</tr>
</tbody>
</table>

4.2 One-Group Anisotropic Scattering

4.2.1 One-Group Pu-239.

One-Energy Group Anisotropic Cross Sections

Table 21 gives the one-group, anisotropic cross sections for two cases of anisotropic scattering. The first cross section set, Pu-239 (a), includes $P_1$ and $P_2$ scattering cross sections, where $|\mu| < 1/3$. The second cross section set, Pu-239 (b), includes the $P_1'$ and $P_2'$ scattering cross sections where $|\mu| > 1/3$. Care must be used to correctly solve this benchmark problem because of the negative scattering for $\mu$ near -1.
Table 21
One-Group Macroscopic Anisotropic Cross Sections (cm\(^{-1}\)) for Pu-239 (\(c=1.40\))

<table>
<thead>
<tr>
<th>Material</th>
<th>(\nu)</th>
<th>(\Sigma_f)</th>
<th>(\Sigma_c)</th>
<th>(\Sigma_{s0})</th>
<th>(\Sigma_{s1})</th>
<th>(\Sigma_{s2})</th>
<th>(\Sigma_t)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pu-239 (a)</td>
<td>2.5</td>
<td>0.266667</td>
<td>0.0</td>
<td>0.733333</td>
<td>0.20</td>
<td>0.075</td>
<td>1.0</td>
<td>1.40</td>
</tr>
<tr>
<td>Pu-239 (b)</td>
<td>2.5</td>
<td>0.266667</td>
<td>0.0</td>
<td>0.733333</td>
<td>0.333333</td>
<td>0.125</td>
<td>1.0</td>
<td>1.40</td>
</tr>
</tbody>
</table>

Infinite Medium (PU-1-1-IN)

Using the cross sections for Pu-239 (a) and Pu-239 (b) in Table 21, \(k_\infty = 2.500000\) (problem 31) with a constant angular and scalar flux everywhere. The anisotropic scattering cross sections do not change \(k_\infty\).

One-Medium Slab Critical Dimensions

The Pu-239 critical dimensions, \(r_c\), for both \(P_1\) and \(P_2\) problems are listed in Table 22.

Table 22
Critical Dimensions, \(r_c\), for One-Group Bare Pu-239 (\(c=1.40\))

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>(r_c) (mfp)</th>
<th>(r_c) (cm)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>PUa-1-1-SL</td>
<td>Slab</td>
<td>0.77032</td>
<td>0.77032</td>
<td>[39]</td>
</tr>
<tr>
<td>33</td>
<td>PUa-1-2-SL</td>
<td>Slab</td>
<td>0.76378</td>
<td>0.76378</td>
<td>[39]</td>
</tr>
<tr>
<td>34</td>
<td>PUB-1-1-SL</td>
<td>Slab</td>
<td>0.79606</td>
<td>0.79606</td>
<td>[39]</td>
</tr>
<tr>
<td>35</td>
<td>PUB 1 2 SL</td>
<td>Slab</td>
<td>0.78396</td>
<td>0.78396</td>
<td>[39]</td>
</tr>
</tbody>
</table>

4.2.2 One-Group U-235

One-Energy Group Anisotropic Cross Sections

Table 23 gives the two sets of one-group, anisotropic cross sections for U-235. Notice that the cross sections are the same as in Table 9 with the addition of \(P_1\) scattering cross sections. The first cross section set, U-235 (a), includes \(P_1\) scattering cross sections, where \(|\mu| < 1/3\). The second cross section set, U-235 (b), includes the \(P_1\) scattering cross sections where \(|\mu| > 1/3\). Care must be used to correctly solve this benchmark problem because of the negative scattering for \(\mu\) near -1.
Table 23
One-Group Macroscopic Anisotropic Cross Sections (cm\(^{-1}\)) for U-235 (\(c=1.30\))

<table>
<thead>
<tr>
<th>Material</th>
<th>(\nu)</th>
<th>(\Sigma_f)</th>
<th>(\Sigma_c)</th>
<th>(\Sigma_{s0})</th>
<th>(\Sigma_{s1})</th>
<th>(\Sigma_t)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>U-235 (a)</td>
<td>2.70</td>
<td>0.065280</td>
<td>0.013056</td>
<td>0.248064</td>
<td>0.042432</td>
<td>0.32640</td>
<td>1.30</td>
</tr>
<tr>
<td>U-235 (b)</td>
<td>2.70</td>
<td>0.065280</td>
<td>0.013056</td>
<td>0.248064</td>
<td>0.212160</td>
<td>0.32640</td>
<td>1.30</td>
</tr>
</tbody>
</table>

Infinite Medium (U-1-1-IN)

Using the cross sections for U-235 (a) and U-235 (b) in Table 23, \(k_\infty = 2.250000\) (problem 11) with a constant angular and scalar flux everywhere. The anisotropic scattering cross sections do not change \(k_\infty\).

One-Medium Slab Critical Dimensions

The U-235 critical dimensions, \(r_c\), for both \(P_1\) problems are listed in Table 24.

Table 24
Critical Dimensions, \(r_c\), for One-Group Bare U-235 (\(c=1.30\))

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>(r_c) (mfp)</th>
<th>(r_c) (cm)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td>Ua-1-1-CY</td>
<td>Cylinder</td>
<td>1.799866479</td>
<td>5.514296811</td>
<td>[40]</td>
</tr>
<tr>
<td>37</td>
<td>Ub-1-1-CY</td>
<td>Cylinder</td>
<td>2.265283130</td>
<td>6.940205668</td>
<td>[40]</td>
</tr>
</tbody>
</table>

4.2.3 One-Group \(U-D_2\)O

One-Energy Group Anisotropic Cross Sections

Table 25 gives the two sets of one-group, anisotropic cross sections for \(U-D_2O\) reactor. Notice that the cross sections are the same as in Table 13 with the addition of \(P_1\) scattering cross sections. The cross sections set for two \(U-D_2O\) cases include \(P_1\) scattering cross sections, where \(|\mu| < 1/3\), and a \(P_1\) case where \(\mu < 0\) and the scattering cross section is negative. Care must be used to correctly solve this benchmark problem because of the negative scattering for \(\mu\) near -1.

Infinite Medium \(UD20a-1-1-IN\), \(UD20b-1-1-IN\), and \(UD20c-1-1-IN\)

Using the cross sections for \(U-D_2O\) (a), \(U-D_2O\) (b), and \(U-D_2O\) (c) in Table 25, \(k_\infty = 1.205587\) (problem 38), 1.227391 (problem 40), and 1.130933 (problem 42), respectively, with a constant angular and scalar flux everywhere. The
Table 25
One-Group Macroscopic Anisotropic Cross Sections (cm\(^{-1}\)) for U-D\(_2\)O Reactor

<table>
<thead>
<tr>
<th>Material</th>
<th>(\nu)</th>
<th>(\Sigma_f)</th>
<th>(\Sigma_{c})</th>
<th>(\Sigma_{s0})</th>
<th>(\Sigma_{s1})</th>
<th>(\Sigma_t)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>U-D(_2)O (a)</td>
<td>1.808381</td>
<td>0.054628</td>
<td>0.027314</td>
<td>0.464338</td>
<td>0.056312624</td>
<td>0.54628</td>
<td>1.0308381</td>
</tr>
<tr>
<td>U-D(_2)O (b)</td>
<td>1.841086</td>
<td>0.054628</td>
<td>0.027314</td>
<td>0.464338</td>
<td>0.112982569</td>
<td>0.54628</td>
<td>1.0341086</td>
</tr>
<tr>
<td>U-D(_2)O (c)</td>
<td>1.6964</td>
<td>0.054628</td>
<td>0.027314</td>
<td>0.464338</td>
<td>-0.27850447</td>
<td>0.54628</td>
<td>1.01964</td>
</tr>
</tbody>
</table>

anisotropic scattering cross sections do not change \(k_{\infty}\).

One-Medium Slab Critical Dimensions

The U-D\(_2\)O critical dimensions, \(r_c\), for the \(P_1\) problems are listed in Table 26.

Table 26
Critical Dimensions, \(r_c\), for One-Group Bare U-D\(_2\)O

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>(r_c) (mfp)</th>
<th>(r_c) (cm)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>39</td>
<td>UD2Oa-1-1-SP</td>
<td>Sphere</td>
<td>10</td>
<td>18.30563081</td>
<td>[22]</td>
</tr>
<tr>
<td>41</td>
<td>UD2Ob-1-1-SP</td>
<td>Sphere</td>
<td>10</td>
<td>18.30563081</td>
<td>[22]</td>
</tr>
<tr>
<td>43</td>
<td>UD2Oc-1-1-SP</td>
<td>Sphere</td>
<td>10</td>
<td>18.30563081</td>
<td>[23]</td>
</tr>
</tbody>
</table>
5 Two-Energy Group Problem Definitions and Results

The isotropic two-energy group cross sections for five bare and two water reflected cases are listed in this section. There are also two linearly anisotropic scattering cross sections sets provided for bare and infinite medium reactors. Unlike the one-group case, there is no flexibility in choosing these values since they are used throughout the literature. The cross sections listed here are similar to Pu-239, [41] U-235,[41] a realistic enriched uranium-aluminum-water assembly [15], a 93% enriched U-235 model of a university research reactor, [15] [44], [45] and a typical large size D2O reactor with low enrichment of U-235.[10], [44], [45] Also included are critical dimensions for a similar uranium research reactor with a water reflector in an infinite lattice.[43] Again, these problems use cross sections that are reasonable representations of the materials described. These cross sections are not general purpose two-group values. The cross sections are used because they are defined in the literature and are intended to be used only to verify algorithm performance and not to predict any actual criticality experiments.

The isotropic neutron cross macroscopic sections (cm\(^{-1}\)) provided for these problems are the total cross section of group \(i\), \(\Sigma_i\), the capture (no neutrons emitted) cross section, \(\Sigma_{cai}\), the within group scattering cross section, \(\Sigma_{isis}\), the group-to-group scattering cross sections, \(\Sigma_{iis_j}\) and \(\Sigma_{sis_j}\), the fission cross section, \(\Sigma_{if}\), the number of neutrons, \(\nu_i\), emitted from each fission in a group, and the fission distribution, \(\chi_i\).

In this paper, the fast energy group is group 2 to be consistent with most of the references. This notation is the reverse of most nuclear engineering textbooks.

The literature solutions are often based on the group transfer cross sections, \(\Sigma_{if}\), given in the references; therefore, the individual cross sections may not be unique. Most references give \((\nu \Sigma_f)_i\) instead of \(\nu_i\) and \(\Sigma_{fi}\). Since both parameters (not the product) may be required by a code for the problem solution, the product \((\nu \Sigma_f)_i\) has been split into \(\nu_i\) and \(\Sigma_{fi}\) preserving their product and \(\Sigma_f\). The infinite slab lattice problems use a slightly unphysical set of cross sections to possibly stress code verification.

The two sets of linearly anisotropic cross sections provided are extensions of the university research reactor and D\(_2\)O cases.[32] The anisotropic scattering component is designated for the in-group and group-to-group scattering cross section by \(\Sigma_{isis}\) and \(\Sigma_{jis_j}\), respectively. Similarly, the isotropic scattering component is designated by \(\Sigma_{iis_0}\) and \(\Sigma_{jis_0}\).

The value for \(k_\infty\) is given for each cross section set. For finite problems, the critical dimension, \(r_c\), is listed in both fast group mean free paths (to indicate the
neutron optical thickness) and in centimeters. For two-media problems, critical
dimensions for the inner multiplying medium and outer reflector thickness are
given in both fast mean free paths and centimeters. The critical dimensions,
$r_c$, and reflector half thicknesses are also given for a water reflected infinite
slab lattice cell. Flux values are given for the university research reactor (a)
(problems 54 and 71) at four spatial points. Angular fluxes can be found in
the dissertation references.

To distinguish between the different URR fissile material cross section sets,
each is labeled with a letter “a,” “b,” “c,” or “d”, respectively. The infinite slab
lattice cell cross sections are similar to the other three cross section sets for
the university research reactor and are labeled with URRd identifiers. The
literature also uses three different H$_2$O reflectors. Their cross sections are also
labeled with a letter “a,” “b,” or “c” in the identifier. URR cross section sets
“d” and “c” have thermal upscattering. All other two-group cross sections have
no thermal upscattering.

A comparison of the critical dimensions for the different geometries behave
as expected; that is, the critical dimension is smallest for the one-dimensional
slab and increases for the cylinder and sphere. This behavior is to be expected
due to the increased leakage with the curvi-linear geometries. The effect of
increased leakage on the critical dimension can be also be seen for the forward
peaked linear anisotropically scattering cases. For the reflected geometries,
the critical dimension decreases with increasing reflector thickness. However,
the critical dimension for the infinite lattice cell increases with the increasing
moderator thickness. Even though this may seem counter-intuitive, it should
be expected because the amount of interaction between the fissile medium and
adjacent cells decreases with increasing moderator half thickness.[43]
5.1 Isotropic Scattering

5.1.1 Two-Group Pu-239

Two-Group Isotropic Cross Sections

Tables 27 and 28 give the two-group, isotropic cross sections for Pu-239.

Table 27
Fast Energy Group Macroscopic Cross Sections (cm\(^{-1}\)) for Pu-239

<table>
<thead>
<tr>
<th>Material</th>
<th>(\nu_2)</th>
<th>(\Sigma_{2f})</th>
<th>(\Sigma_{2c})</th>
<th>(\Sigma_{22s})</th>
<th>(\Sigma_{12s})</th>
<th>(\Sigma_2)</th>
<th>(\chi_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pu-239</td>
<td>3.10</td>
<td>0.0936</td>
<td>0.00480</td>
<td>0.0792</td>
<td>0.0432</td>
<td>0.2208</td>
<td>0.575</td>
</tr>
</tbody>
</table>

Table 28
Slow Energy Group Macroscopic Cross Sections (cm\(^{-1}\)) for Pu-239

<table>
<thead>
<tr>
<th>Material</th>
<th>(\nu_1)</th>
<th>(\Sigma_{1f})</th>
<th>(\Sigma_{1c})</th>
<th>(\Sigma_{11s})</th>
<th>(\Sigma_{21s})</th>
<th>(\Sigma_1)</th>
<th>(\chi_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pu-239</td>
<td>2.93</td>
<td>0.08544</td>
<td>0.0144</td>
<td>0.23616</td>
<td>0.0</td>
<td>0.3360</td>
<td>0.425</td>
</tr>
</tbody>
</table>

Infinite Medium (PU-2-O-IN)

Using the two-group isotropic Pu-239 cross section set from Tables 27 and 28, \(k_\infty = 2.683767\) (problem 44) with a constant group angular and scalar flux and a group 2 to group 1 flux ratio = 0.675229.

One-Medium Slab and Sphere Critical Dimensions

The critical dimensions, \(r_c\), are listed in Table 29.

Table 29
Critical Dimensions, \(r_c\), for Two-Group Bare Pu-239

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>(r_c) (mfp)</th>
<th>(r_c) (cm)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>P'U-2-O-SL</td>
<td>Slab</td>
<td>0.396469</td>
<td>1.795602</td>
<td>[15], [44], [45]</td>
</tr>
<tr>
<td>46</td>
<td>P'U-2-O-SP</td>
<td>Sphere</td>
<td>1.15513</td>
<td>5.231567</td>
<td>[15], [44], [45]</td>
</tr>
</tbody>
</table>
5.1.2 Two-Group U-235.

Two-Group Cross Sections

Tables 30 and 31 give the two-group, isotropic cross sections for U-235.

**Table 30**
Fast Energy Group Macroscopic Cross Sections (cm$^{-1}$) for U-235

<table>
<thead>
<tr>
<th>Material</th>
<th>$\nu_2$</th>
<th>$\Sigma_{2f}$</th>
<th>$\Sigma_{2c}$</th>
<th>$\Sigma_{22s}$</th>
<th>$\Sigma_{12s}$</th>
<th>$\Sigma_2$</th>
<th>$\chi_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>U-235</td>
<td>2.70</td>
<td>0.06192</td>
<td>0.00384</td>
<td>0.078240</td>
<td>0.0720</td>
<td>0.2160</td>
<td>0.575</td>
</tr>
</tbody>
</table>

**Table 31**
Slow Energy Group Macroscopic Cross Sections (cm$^{-1}$) for U-235

<table>
<thead>
<tr>
<th>Material</th>
<th>$\nu_1$</th>
<th>$\Sigma_{1f}$</th>
<th>$\Sigma_{1c}$</th>
<th>$\Sigma_{11s}$</th>
<th>$\Sigma_{21s}$</th>
<th>$\Sigma_1$</th>
<th>$\chi_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>U-235</td>
<td>2.50</td>
<td>0.06912</td>
<td>0.01344</td>
<td>0.26304</td>
<td>0.0</td>
<td>0.3456</td>
<td>0.425</td>
</tr>
</tbody>
</table>

Infinite Medium (U-2-0-IN)

Using the two-group U-235 cross section set from Tables 30 and 31, $k_{\infty} = 2.216349$ (problem 47) with a constant group angular and scalar flux and the group 2 to group 1 flux ratio = 0.474967.

One-Medium Slab and Sphere Critical Dimensions

The critical dimensions, $r_c$, are listed in Table 32.

**Table 32**
Critical Dimension, $r_c$, for Two-Group Bare U-235

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>$r_c$ (mfp)</th>
<th>$r_c$ (cm)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>48</td>
<td>U-2-0-SL</td>
<td>Slab</td>
<td>0.649377</td>
<td>3.006375</td>
<td>[15], [44], [45]</td>
</tr>
<tr>
<td>49</td>
<td>U-2-0-SP</td>
<td>Sphere</td>
<td>1.70844</td>
<td>7.909444</td>
<td>[15]</td>
</tr>
</tbody>
</table>
Criticality code verification

5.1.3 Two-Group Uranium-Aluminum-Water Assembly.

Two-Group Isotropic Cross Sections

Tables 33 and 34 give the two-group, isotropic cross sections for the uranium, aluminum, and water assembly.

Table 33
Fast Energy Group Macroscopic Cross Sections (cm\(^{-1}\)) for U-Al

<table>
<thead>
<tr>
<th>Material</th>
<th>(\nu_2)</th>
<th>(\Sigma_{2f})</th>
<th>(\Sigma_{2c})</th>
<th>(\Sigma_{22s})</th>
<th>(\Sigma_{12s})</th>
<th>(\Sigma_2)</th>
<th>(\chi_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>U-Al</td>
<td>0.0</td>
<td>0.0</td>
<td>0.000217</td>
<td>0.247516</td>
<td>0.020432</td>
<td>0.268165</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 34
Slow Energy Group Macroscopic Cross Sections (cm\(^{-1}\)) for U-Al

<table>
<thead>
<tr>
<th>Material</th>
<th>(\nu_1)</th>
<th>(\Sigma_{1f})</th>
<th>(\Sigma_{1c})</th>
<th>(\Sigma_{11s})</th>
<th>(\Sigma_{21s})</th>
<th>(\Sigma_1)</th>
<th>(\chi_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>U-Al</td>
<td>2.830023</td>
<td>0.060706</td>
<td>0.003143</td>
<td>1.213127</td>
<td>0.0</td>
<td>1.276976</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Infinite Medium (UAL-2-0-IN)

With the two-group cross section set from Tables 33 and 34, \(k_\infty = 2.662437\) (problem 50) and the group 2 to group 1 flux ratio = 3.124951.

One-Medium Slab and Sphere Critical Dimensions

Using the cross sections given in Tables 33 and 34, the critical dimensions, \(r_c\), are given in Table 35.

Table 35
Critical Dimensions, \(r_c\), for Two-Group Uranium-Aluminum-Water Assembly

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>(r_c) (mfp)</th>
<th>(r_c) (cm)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>51</td>
<td>UAL-2-0-SL</td>
<td>Slab</td>
<td>2.09994</td>
<td>7.830776</td>
<td>[15], [44], [45]</td>
</tr>
<tr>
<td>52</td>
<td>UAL-2-0-SP</td>
<td>Sphere</td>
<td>4.73786</td>
<td>17.66770</td>
<td>[15]</td>
</tr>
</tbody>
</table>
5.1.4 Two-Group Uranium Research Reactor.

Two-Group Isotropic Cross Sections

The cross sections for the one-medium (a), two-media (b and c), and infinite slab lattice (d) cases are different and are therefore listed separately. Tables 36 and 37 gives the two group, one medium, isotropic cross sections for the 93% enriched uranium bare university research reactor.

Table 36
Fast Energy Group Macroscopic Cross Sections (cm\(^{-1}\)) for Research Reactor (a)

<table>
<thead>
<tr>
<th>Material</th>
<th>(\nu_2)</th>
<th>(\Sigma_{2f})</th>
<th>(\Sigma_{2c})</th>
<th>(\Sigma_{22s})</th>
<th>(\Sigma_{12s})</th>
<th>(\Sigma_2)</th>
<th>(\chi_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Research Reactor (a)</td>
<td>2.50</td>
<td>0.0010484</td>
<td>0.0010046</td>
<td>0.62568</td>
<td>0.029227</td>
<td>0.65696</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 37
Slow Energy Group Macroscopic Cross Sections (cm\(^{-1}\)) for Research Reactor (a)

<table>
<thead>
<tr>
<th>Material</th>
<th>(\nu_1)</th>
<th>(\Sigma_{1f})</th>
<th>(\Sigma_{1c})</th>
<th>(\Sigma_{21s})</th>
<th>(\Sigma_{21s})</th>
<th>(\Sigma_1)</th>
<th>(\chi_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Research Reactor (a)</td>
<td>2.50</td>
<td>0.050632</td>
<td>0.025788</td>
<td>2.44383</td>
<td>0.0</td>
<td>2.52025</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Infinite Medium (URRa-2-0-IN)

The test set uses the two-group enriched U-235 cross section set for the research reactor in Tables 36 and 37 with \(k_\infty = 1.631452\) (problem 53) and the group 2 to group 1 flux ratio = 2.614706.

One-Medium Slab and Sphere Critical Dimensions

The critical dimensions, \(r_c\), are listed in Table 38.

Table 38
Critical Dimensions, \(r_c\), for Two-Group Bare Research Reactor (a)

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>(r_c) (mfp)</th>
<th>(r_c) (cm)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>54</td>
<td>URRa-2-0-SL</td>
<td>Slab</td>
<td>4.97112</td>
<td>7.566853</td>
<td>[15],[44],[45]</td>
</tr>
<tr>
<td>55</td>
<td>URRa-2-0-SP</td>
<td>Sphere</td>
<td>10.5441</td>
<td>16.049836</td>
<td>[15]</td>
</tr>
</tbody>
</table>

One-Medium Slab Scalar Neutron Fluxes

Table 39 gives the normalized scalar neutron flux for the two-group bare research reactor (a) at four spatial points [45]. All values are normalized with the fast group flux at the center.
Table 39
Normalized Scalar Fluxes for Two-Group Bare Research Reactor (a)

<table>
<thead>
<tr>
<th>Problem Identifier</th>
<th>Geometry</th>
<th>Energy Group</th>
<th>r/r_ε = 0.241394</th>
<th>r/r_ε = 0.502905</th>
<th>r/r_ε = 0.744300</th>
<th>r/r_ε = 1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>54 URRa-2-0-SL</td>
<td>Slab</td>
<td>Fast</td>
<td>0.943363</td>
<td>0.791973</td>
<td>0.594012</td>
<td>0.147095</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Slow</td>
<td>0.340124</td>
<td>0.273056</td>
<td>0.173845</td>
<td>0.0212324</td>
</tr>
</tbody>
</table>

Two-Media Cross Sections for Slab Geometry

The cross sections for the two-media problems for the uranium research reactor are given for two different multiplying media with one nonmultiplying reflector. The two multiplying materials are labeled (b) and (c), respectively. The multiplying region consists of an H2O + U-235 mixture surrounded by an H2O reflector. The results in the literature for case (c) only include the infinite water reflector. The cross sections are given in Tables 40 and 41. Notice that this problem allows thermal upscattering in both multiplying and nonmultiplying regions.

Table 40
Fast Energy Group Macroscopic Cross Sections (cm⁻¹) for Research Reactor (b), (c) and H2O Reflector (a)

<table>
<thead>
<tr>
<th>Material</th>
<th>ν₂</th>
<th>Σ₂f</th>
<th>Σ₂c</th>
<th>Σ₂₂s</th>
<th>Σ₁₂s</th>
<th>Σ₂</th>
<th>χ₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>Research Reactor (b)</td>
<td>2.50</td>
<td>0.000836</td>
<td>0.001104</td>
<td>0.83892</td>
<td>0.04635</td>
<td>0.88721</td>
<td>1.0</td>
</tr>
<tr>
<td>Research Reactor (c)</td>
<td>2.50</td>
<td>0.001648</td>
<td>0.001472</td>
<td>0.83807</td>
<td>0.04536</td>
<td>0.88655</td>
<td>1.0</td>
</tr>
<tr>
<td>H2O (a) (refl)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.00074</td>
<td>0.83975</td>
<td>0.04749</td>
<td>0.88798</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table 41
Slow Energy Group Macroscopic Cross Sections (cm⁻¹) for Research Reactor (b), (c) and H2O Reflector (a)

<table>
<thead>
<tr>
<th>Material</th>
<th>ν₁</th>
<th>Σ₁f</th>
<th>Σ₁c</th>
<th>Σ₁₁s</th>
<th>Σ₂₁s</th>
<th>Σ₁</th>
<th>χ₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>Research Reactor (b)</td>
<td>2.50</td>
<td>0.029564</td>
<td>0.024069</td>
<td>2.9183</td>
<td>0.000767</td>
<td>2.9727</td>
<td>0.0</td>
</tr>
<tr>
<td>Research Reactor (c)</td>
<td>2.50</td>
<td>0.057296</td>
<td>0.029244</td>
<td>2.8751</td>
<td>0.00116</td>
<td>2.9628</td>
<td>0.0</td>
</tr>
<tr>
<td>H2O (a) (refl)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.018564</td>
<td>2.9676</td>
<td>0.000336</td>
<td>2.9865</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Infinite Medium (URRb-2-0-IN and URRc-2-0-IN)

Using the two-group cross section set for the research reactor (b) from Tables 40 and 41, k_∞ = 1.365821 (problem 56) with a constant group angular and scalar flux and the group 2 to group 1 flux ratio = 1.173679. Using the two-group cross section set for research reactor (c) from Tables 40 and 41, k_∞ = 1.633380 (problem 57)
with a constant group angular and scalar flux and the group 2 to group 1 flux ratio $= 1.933422$.

Two-Media Slab Critical Dimensions

Using the cross sections in Tables 40 and 41 for the $\text{H}_2\text{O} + \text{U}-235$ research reactor and the $\text{H}_2\text{O}$ reflector, the critical dimensions are given in Table 42. The mfp results use the group 2 total macroscopic cross section of region $i$ to obtain the dimensions in cm.

Table 42
Critical Dimensions for Two-Group Research Reactor (b),(c) with $\text{H}_2\text{O}$ Reflector (a)

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>$\text{U}-235, r_c$</th>
<th>$\text{H}_2\text{O}$ Width</th>
<th>$\text{U}-235 + \text{H}_2\text{O}$ Width</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>58</td>
<td>URRb-H2Oa(1)-2-0-SL</td>
<td>Slab (mfp)</td>
<td>5.94147</td>
<td>1</td>
<td>7.822954</td>
<td>[46]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(cm)</td>
<td>6.696802</td>
<td>1.126152</td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>URRb-H2Oa(5)-2-0-SL</td>
<td>Slab (mfp)</td>
<td>4.31485</td>
<td>5</td>
<td>10.494149</td>
<td>[46]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(cm)</td>
<td>4.863392</td>
<td>5.630757</td>
<td></td>
<td></td>
</tr>
<tr>
<td>61</td>
<td>URRb-H2Oa(IN)-2-0-SL</td>
<td>Slab (mfp)</td>
<td>4.15767</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>[46]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(cm)</td>
<td>4.686230</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td></td>
</tr>
<tr>
<td>61</td>
<td>URRc-H2Oa(IN)-2-0-SL</td>
<td>Slab (mfp)</td>
<td>2.1826</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>[46]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(cm)</td>
<td>2.461903</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td></td>
</tr>
</tbody>
</table>
Two-Media Cross Sections for Infinite Slab Lattice Cell

The two-media cross sections are given in Tables 43 and 44 for a similar uranium enriched university research reactor. The $\nu_2$ is slightly unphysical to stress criticality codes. Two different reflector materials are also given in Tables 43 and 44. The problems that use these cross sections are for an infinite slab lattice cell.

Table 43
Fast Group Macroscopic Cross Sections ($cm^{-1}$) for Research Reactor(d) and H$_2$O Reflector (b), (c)

<table>
<thead>
<tr>
<th>Material</th>
<th>$\nu_2$</th>
<th>$\Sigma_{2f}$</th>
<th>$\Sigma_{2c}$</th>
<th>$\Sigma_{22a}$</th>
<th>$\Sigma_{12a}$</th>
<th>$\Sigma_2$</th>
<th>$\chi_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Research Reactor (d)</td>
<td>1.004</td>
<td>0.61475</td>
<td>0.0019662</td>
<td>0.0</td>
<td>0.0342008</td>
<td>0.650917</td>
<td>1.0</td>
</tr>
<tr>
<td>H$_2$O (b) (refl)</td>
<td>0.0</td>
<td>0.0</td>
<td>8.480293x10$^{-6}$</td>
<td>0.1096742149</td>
<td>0.001000592707</td>
<td>0.1106032900</td>
<td>0.0</td>
</tr>
<tr>
<td>H$_2$O (c) (refl)</td>
<td>0.0</td>
<td>0.0</td>
<td>4.97229x10$^{-4}$</td>
<td>1.226381244</td>
<td>0.1046395340</td>
<td>1.331518007</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table 44
Slow Group Macroscopic Cross Sections ($cm^{-1}$) for Research Reactor(d) and H$_2$O Reflector (b), (c)

<table>
<thead>
<tr>
<th>Material</th>
<th>$\nu_1$</th>
<th>$\Sigma_{1f}$</th>
<th>$\Sigma_{1c}$</th>
<th>$\Sigma_{11s}$</th>
<th>$\Sigma_{21s}$</th>
<th>$\Sigma_1$</th>
<th>$\chi_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Research Reactor (d)</td>
<td>2.50</td>
<td>0.045704</td>
<td>0.023496</td>
<td>2.06880</td>
<td>0.0</td>
<td>2.13800</td>
<td>0.0</td>
</tr>
<tr>
<td>H$_2$O (b) (refl)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.00016</td>
<td>0.36339</td>
<td>0.0</td>
<td>0.36355</td>
<td>0.0</td>
</tr>
<tr>
<td>H$_2$O (c) (refl)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0188</td>
<td>4.35470</td>
<td>0.0</td>
<td>4.37350</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Infinite Medium (URRd-2.0-IN)

The test set uses the two-group cross section set for the research reactor in Tables 43 and 44 with $k_\infty = 1.034970$ (problem 62) and the group 2 to group 1 flux ratio = 2.023344.

Two-Media Infinite Slab Lattice Cell Critical Dimensions

Using the cross sections in Tables 43 and 44 for the enriched uranium research reactor with a H$_2$O reflector, the critical dimensions for an infinite slab lattice cell as shown in Figure 2 are given in Table 45. Because this is an infinite slab lattice cell with reflecting outer boundaries, notice that the moderator half thickness is given.
Table 45
Critical Dimensions for Two-Group Infinite Slab Lattice Cell and H₂O Reflector (b), (c)

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier/Geometry</th>
<th>U-235, r_c</th>
<th>H₂O Width</th>
<th>U-235+H₂O Width</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>63</td>
<td>URRd-H₂Ob(l)-2-0-ISLC Inf. Slab (mfp)</td>
<td>0.02142</td>
<td>1</td>
<td>9.034797</td>
<td>9.067695</td>
</tr>
<tr>
<td></td>
<td>Lat. Cell (cm)</td>
<td>0.0329074</td>
<td>9.034797</td>
<td>9.067695</td>
<td>[43]</td>
</tr>
<tr>
<td>64</td>
<td>URRd-H₂Ob(10)-2-0-ISLC Inf. Slab (mfp)</td>
<td>0.29951</td>
<td>10</td>
<td>90.347875</td>
<td>90.808010</td>
</tr>
<tr>
<td></td>
<td>Lat. Cell (cm)</td>
<td>0.460135</td>
<td>90.347875</td>
<td>90.808010</td>
<td>[43]</td>
</tr>
<tr>
<td>65</td>
<td>URRd-H₂Oc(l)-2-0-ISLC Inf. Slab (mfp)</td>
<td>0.22197</td>
<td>1</td>
<td>0.751023</td>
<td>1.092034</td>
</tr>
<tr>
<td></td>
<td>Lat. Cell (cm)</td>
<td>0.341011</td>
<td>0.751023</td>
<td>1.092034</td>
<td>[43]</td>
</tr>
<tr>
<td>66</td>
<td>URRd-H₂Oc(10)-2-0-ISLC Inf. Slab (mfp)</td>
<td>1.7699</td>
<td>10</td>
<td>7.510225</td>
<td>10.229312</td>
</tr>
<tr>
<td></td>
<td>Lat. Cell (cm)</td>
<td>2.719087</td>
<td>7.510225</td>
<td>10.229312</td>
<td>[43]</td>
</tr>
</tbody>
</table>

5.1.5 Two-Group U-D₂O Reactor

Two-Group Isotropic Cross Sections

Tables 46 and 47 give the two-group, isotropic cross sections for the uranium-D₂O system.

Table 46
Fast Energy Group Macroscopic Cross Sections (cm⁻¹) for U-D₂O

<table>
<thead>
<tr>
<th>Material</th>
<th>μ₂</th>
<th>Σ₂f</th>
<th>Σ₂c</th>
<th>Σ₂g2</th>
<th>Σ₂g2s</th>
<th>Σ₂g2s</th>
<th>Σ₂g2s</th>
<th>Σ₂g2s</th>
<th>Σ₂g2s</th>
<th>Σ₂g2s</th>
<th>Σ₂g2s</th>
<th>Σ₂g2s</th>
<th>Σ₂g2s</th>
<th>Σ₂g2s</th>
<th>Σ₂g2s</th>
</tr>
</thead>
<tbody>
<tr>
<td>U-D₂O</td>
<td>2.50</td>
<td>0.002817</td>
<td>0.0087078</td>
<td>0.31980</td>
<td>0.0045552</td>
<td>0.33588</td>
<td>1.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 47
Slow Energy Group Macroscopic Cross Sections (cm⁻¹) for U-D₂O

| Material | μ₁ | Σ₁f | Σ₁c | Σ₁s1 | Σ₁s1 | Σ₁s1 | Σ₁s1 | Σ₁s1 | Σ₁s1 | Σ₁s1 | Σ₁s1 | Σ₁s1 | Σ₁s1 | Σ₁s1 | Σ₁s1 |
|----------|----|-----|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| U-D₂O    | 2.50 | 0.097 | 0.02518 | 0.42410 | 0.0 | 0.54628 | 0.0 |

Infinite Medium (UD2O-2-0-IN)

The test set uses the two-group U-D₂O cross section set from Tables 46 and 47 with \( k_\infty = 1.000221 \) (problem 67) and the group 2 to group 1 flux ratio = 26.822093.

One-Medium Slab and Sphere Critical Dimensions

The critical dimensions, \( r_c \), are listed in Table 48.
Table 48
Critical Dimension, \( r_c \), for Two-Group D\(_2\)O System

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>( r_c ) (mfp)</th>
<th>( r_c ) (cm)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>68</td>
<td>UD2O-2-0-SL</td>
<td>Slab</td>
<td>284.367</td>
<td>846.632726</td>
<td>[15], [44], [45]</td>
</tr>
<tr>
<td>69</td>
<td>UD2O-2-0-SP</td>
<td>Sphere</td>
<td>569.430</td>
<td>1695.337621</td>
<td>[15]</td>
</tr>
</tbody>
</table>

5.2 Linearly Anisotropic Scattering

The anisotropic scattering cross sections for the enriched U-235 research reactor and U-D\(_2\)O reactor cases are the same as the isotropic set with the addition of the anisotropic cross sections, \( \Sigma_{22s1} \), \( \Sigma_{12s1} \), and \( \Sigma_{11s1} \).

5.2.1 Two-Group Uranium Research Reactor.

Two-Group Anisotropic Macroscopic Cross Sections

Tables 49 and 50 gives the two-group, linearly anisotropic cross sections for the research reactor. **Care must be used to correctly solve this benchmark problem because of the negative scattering for \( \mu \) near -1.**

Table 49
Fast Group Cross Sections for Linearly Anisotropic Scattering (cm\(^{-1}\)) Research Reactor (a)

<table>
<thead>
<tr>
<th>Material</th>
<th>( \nu_f )</th>
<th>( \Sigma_{2f} )</th>
<th>( \Sigma_{2s} )</th>
<th>( \Sigma_{2s0} )</th>
<th>( \Sigma_{2s1} )</th>
<th>( \Sigma_{12s0} )</th>
<th>( \Sigma_{12s1} )</th>
<th>( \Sigma_{2} )</th>
<th>( \chi_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Research Reactor (a)</td>
<td>2.50</td>
<td>0.0010484</td>
<td>0.0010046</td>
<td>0.62568</td>
<td>0.27459</td>
<td>0.029227</td>
<td>0.0075737</td>
<td>0.65696</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 50
Slow Group Cross Sections for Linearly Anisotropic Scattering (cm\(^{-1}\)) Research Reactor (a)

<table>
<thead>
<tr>
<th>Material</th>
<th>( \nu_f )</th>
<th>( \Sigma_{1f} )</th>
<th>( \Sigma_{1s} )</th>
<th>( \Sigma_{1s0} )</th>
<th>( \Sigma_{1s1} )</th>
<th>( \Sigma_{21s} )</th>
<th>( \Sigma_{1} )</th>
<th>( \chi_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Research Reactor (a)</td>
<td>2.50</td>
<td>0.050632</td>
<td>0.025788</td>
<td>2.44383</td>
<td>0.83318</td>
<td>0.0</td>
<td>2.52025</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Infinite Medium (URRa-2-1-IN)

The test set uses the two-group enriched U-235 cross section set from Tables 49 and 50 with \( k_{\infty} = 1.631452 \) (problem 70) and the group 2 to group 1 flux ratio = 2.614706.

One-Medium Slab Critical Dimension

The critical dimensions are listed in Table 51.
Table 51
Critical Dimension, $r_c$, for Two-Group Linearly Anisotropic Scattering Research Reactor (a)

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>$r_c$ (mfp)</th>
<th>$r_c$ (cm)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>71</td>
<td>URRa-2-1-SL</td>
<td>Slab</td>
<td>6.2384</td>
<td>9.4959</td>
<td>[34]</td>
</tr>
</tbody>
</table>

One-Medium Slab Scalar Neutron Fluxes

Table 52 gives the normalized scalar neutron flux for the two-group bare research reactor (a) with linearly anisotropic scattering at four spatial points. All values are normalized with the fast group flux at the center.

Table 52
Normalized Scalar Fluxes for Two-Group Bare Research Reactor (a)

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>Energy Group</th>
<th>$r/r_c = 0.20$</th>
<th>$r/r_c = 0.50$</th>
<th>$r/r_c = 0.80$</th>
<th>$r/r_c = 1.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>71</td>
<td>URRa-2-1-SL</td>
<td>Slab</td>
<td>Fast</td>
<td>0.963873</td>
<td>0.781389</td>
<td>0.472787</td>
<td>0.189578</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Slow</td>
<td>0.349006</td>
<td>0.280870</td>
<td>0.157376</td>
<td>0.0277639</td>
</tr>
</tbody>
</table>

5.2.2 Two-Group U-D$_2$O Reactor.

Two-Group Anisotropic Macroscopic Cross Sections

Tables 53 and 54 gives the two-group, linearly anisotropic cross sections for the U-D$_2$O system.

Table 53
Fast Energy Group Cross Sections for Linearly Anisotropic Scattering (cm$^{-1}$) for U-D$_2$O

<table>
<thead>
<tr>
<th>Material</th>
<th>$\nu_1$</th>
<th>$\Sigma_{if}$</th>
<th>$\Sigma_{if}$</th>
<th>$\Sigma_{2280}$</th>
<th>$\Sigma_{2281}$</th>
<th>$\Sigma_{1280}$</th>
<th>$\Sigma_{1281}$</th>
<th>$\Sigma_2$</th>
<th>$\chi_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>D$_2$O</td>
<td>2.50</td>
<td>0.0028172</td>
<td>0.0087078</td>
<td>0.31980</td>
<td>0.06694</td>
<td>0.004555</td>
<td>-0.0003972</td>
<td>0.33588</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 54
Slow Energy Group Cross Sections for Linearly Anisotropic Scattering (cm$^{-1}$) for U-D$_2$O

<table>
<thead>
<tr>
<th>Material</th>
<th>$\nu_1$</th>
<th>$\Sigma_{1f}$</th>
<th>$\Sigma_{1c}$</th>
<th>$\Sigma_{1180}$</th>
<th>$\Sigma_{1181}$</th>
<th>$\Sigma_{21}$</th>
<th>$\Sigma_1$</th>
<th>$\chi_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>D$_2$O</td>
<td>2.50</td>
<td>0.097</td>
<td>0.02518</td>
<td>0.42410</td>
<td>0.02439</td>
<td>0.0</td>
<td>0.34028</td>
<td>u.u</td>
</tr>
</tbody>
</table>

Infinite Medium (UD2O-2-1-IN)

The test set uses the two-group linearly anisotropic D$_2$O cross section set from Tables 53 and 54 with $k_\infty = 1.000227$ (problem 72) and the group 2 to group 1 flux
ratio = 26.823271.

One-Medium Slab Critical Dimension

The critical dimension, \( r_c \), is listed in Table 55.

Table 55

<table>
<thead>
<tr>
<th>Problem</th>
<th>Identifier</th>
<th>Geometry</th>
<th>( r_c ) (mfp)</th>
<th>( r_c ) (cm)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>73</td>
<td>UD2O-2-1-SL</td>
<td>Slab</td>
<td>312.18</td>
<td>929.45</td>
<td>[34]</td>
</tr>
</tbody>
</table>

6 Three-Energy Group Problem Definitions and Results

A three-energy group isotropic infinite medium problem is defined in this section. The derivation appears in Appendix A. This problem assumes no thermal upscattering and no fission neutrons born in the slowest energy group.

The fast energy group is group 3 to be consistent with most of the references. Again, this notation is the reverse of most nuclear engineering textbooks.

The cross sections listed here are similar to the uranium university research reactors. Again, this problem uses cross sections that are reasonable representations of the materials described and are not general purpose values. The cross sections are intended to be used to verify algorithm performance and not to predict criticality experiments. The cross sections are from [47] and are derived in Appendix A.

Infinite Medium (URR-3-0-IN )

Using the three-group cross section set from Tables 56, 57, and 58, \( k_{\infty} = 1.600000 \) (problem 74) with a constant group angular and scalar flux and the group 2 to group 3 flux ratio = 0.480, the group 1 to group 2 flux ratio = 0.3125, and the group 1 to group 3 flux ratio = 0.150.
Table 56
Fast Energy Group Macroscopic Cross Sections (cm\(^{-1}\)) for Research Reactor

<table>
<thead>
<tr>
<th>Material</th>
<th>(\nu_{\text{f}})</th>
<th>(\Sigma_{2f})</th>
<th>(\Sigma_{3f})</th>
<th>(\Sigma_{33s})</th>
<th>(\Sigma_{23s})</th>
<th>(\Sigma_{13s})</th>
<th>(\Sigma_{3})</th>
<th>(\chi_{3})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Research Reactor</td>
<td>3.0</td>
<td>0.006</td>
<td>0.006</td>
<td>0.024</td>
<td>0.171</td>
<td>0.033</td>
<td>0.240</td>
<td>0.96</td>
</tr>
</tbody>
</table>

Table 57
Middle Energy Group Macroscopic Cross Sections (cm\(^{-1}\)) for Research Reactor

<table>
<thead>
<tr>
<th>Material</th>
<th>(\nu_{\text{f}})</th>
<th>(\Sigma_{2f})</th>
<th>(\Sigma_{2c})</th>
<th>(\Sigma_{22s})</th>
<th>(\Sigma_{32s})</th>
<th>(\Sigma_{12s})</th>
<th>(\Sigma_{2})</th>
<th>(\chi_{2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Research Reactor</td>
<td>2.50</td>
<td>0.060</td>
<td>0.040</td>
<td>0.60</td>
<td>0.0</td>
<td>0.275</td>
<td>0.975</td>
<td>0.04</td>
</tr>
</tbody>
</table>

Table 58
Slow Energy Group Macroscopic Cross Sections (cm\(^{-1}\)) for Research Reactor

<table>
<thead>
<tr>
<th>Material</th>
<th>(\nu_{\text{f}})</th>
<th>(\Sigma_{1f})</th>
<th>(\Sigma_{1c})</th>
<th>(\Sigma_{11s})</th>
<th>(\Sigma_{21s})</th>
<th>(\Sigma_{31s})</th>
<th>(\Sigma_{1})</th>
<th>(\chi_{1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Research Reactor</td>
<td>2.0</td>
<td>0.90</td>
<td>0.20</td>
<td>2.0</td>
<td>0.0</td>
<td>0.0</td>
<td>3.10</td>
<td>0.0</td>
</tr>
</tbody>
</table>

7 Six-Energy Group Problem Definitions and Results

A six-energy group isotropic infinite medium problem comprised of two coupled three-energy group cross sections used in URR-3-0-IN is defined in this section. This test problem defines a six group cross section set \([48]\) such that energy groups 6 and 1, 5 and 2, and 4 and 3 are equivalent. The top three groups are decoupled from the lower three groups except for the fission distribution, \(\chi_{1}\), which affects energy groups 6, 5, 2, and 1. Energy group 6 (group 1) scatters to groups 5 and 4 (groups 2 and 3). Energy group 5 (group 2) scatters to group 4 (group 3). Energy group 4 (group 3) self-scatters only. Since groups 1, 2, and 3 are upscatter equivalents of groups 6, 5, and 4, respectively, this problem should only be used with codes that allow for thermal upscattering.

Infinite Medium (URR-6-0-IN)

Since this problem is comprised of problem URR-3-0-IN cross sections with modified \(\chi_{1}\) values, the final \(k_{\infty}\) value and flux ratios will not change. Using the six-group cross section set from Tables 59, 60, 61 62, 63, and 64, \(k_{\infty} = 1.600000\) (problem 75) with a constant angular and scalar flux in each group. The group 5 to group 6 and group 2 to group 1 flux ratio = 0.480, the group 4 to group 5 and group 3 to group 2 flux ratio = 0.3125, and a group 4 to group 6 and group 3 to group 1 flux ratio = 0.150. These ratios are the same as in the three-group problem.
Table 59
Fast Energy Group 6 Macroscopic Cross Sections (cm\(^{-1}\)) for Research Reactor

<table>
<thead>
<tr>
<th>Material</th>
<th>(\nu_6)</th>
<th>(\Sigma_{6f})</th>
<th>(\Sigma_{6c})</th>
<th>(\Sigma_{6s})</th>
<th>(\Sigma_{6s})</th>
<th>(\Sigma_{6s})</th>
<th>(\Sigma_6)</th>
<th>(\chi_6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Research Reactor</td>
<td>3.0</td>
<td>0.006</td>
<td>0.006</td>
<td>0.024</td>
<td>0.171</td>
<td>0.033</td>
<td>0.240</td>
<td>0.45</td>
</tr>
</tbody>
</table>

Table 60
Energy Group 5 Macroscopic Cross Sections (cm\(^{-1}\)) for Research Reactor

<table>
<thead>
<tr>
<th>Material</th>
<th>(\nu_5)</th>
<th>(\Sigma_{5f})</th>
<th>(\Sigma_{5c})</th>
<th>(\Sigma_{5s})</th>
<th>(\Sigma_{5s})</th>
<th>(\Sigma_{5s})</th>
<th>(\Sigma_5)</th>
<th>(\chi_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Research Reactor</td>
<td>2.50</td>
<td>0.060</td>
<td>0.040</td>
<td>0.60</td>
<td>0.0</td>
<td>0.275</td>
<td>0.975</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Table 61
Energy Group 4 Macroscopic Cross Sections (cm\(^{-1}\)) for Research Reactor

<table>
<thead>
<tr>
<th>Material</th>
<th>(\nu_4)</th>
<th>(\Sigma_{4f})</th>
<th>(\Sigma_{4c})</th>
<th>(\Sigma_{4s})</th>
<th>(\Sigma_{4s})</th>
<th>(\Sigma_{4s})</th>
<th>(\Sigma_4)</th>
<th>(\chi_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Research Reactor</td>
<td>2.0</td>
<td>0.90</td>
<td>0.20</td>
<td>2.0</td>
<td>0.0</td>
<td>3.10</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table 62
Energy Group 3 Macroscopic Cross Sections (cm\(^{-1}\)) for Research Reactor

<table>
<thead>
<tr>
<th>Material</th>
<th>(\nu_3)</th>
<th>(\Sigma_{3f})</th>
<th>(\Sigma_{3c})</th>
<th>(\Sigma_{3s})</th>
<th>(\Sigma_{3s})</th>
<th>(\Sigma_{3s})</th>
<th>(\Sigma_3)</th>
<th>(\chi_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Research Reactor</td>
<td>2.0</td>
<td>0.90</td>
<td>0.20</td>
<td>2.0</td>
<td>0.0</td>
<td>3.10</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table 63
Energy Group 2 Macroscopic Cross Sections (cm\(^{-1}\)) for Research Reactor

<table>
<thead>
<tr>
<th>Material</th>
<th>(\nu_2)</th>
<th>(\Sigma_{2f})</th>
<th>(\Sigma_{2c})</th>
<th>(\Sigma_{2s})</th>
<th>(\Sigma_{2s})</th>
<th>(\Sigma_{2s})</th>
<th>(\Sigma_2)</th>
<th>(\chi_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Research Reactor</td>
<td>2.50</td>
<td>0.060</td>
<td>0.040</td>
<td>0.60</td>
<td>0.0</td>
<td>0.275</td>
<td>0.975</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Table 64
Slow Energy Group 1 Macroscopic Cross Sections (cm\(^{-1}\)) for Research Reactor

<table>
<thead>
<tr>
<th>Material</th>
<th>(\nu_1)</th>
<th>(\Sigma_{1f})</th>
<th>(\Sigma_{1c})</th>
<th>(\Sigma_{1s})</th>
<th>(\Sigma_{1s})</th>
<th>(\Sigma_{1s})</th>
<th>(\Sigma_1)</th>
<th>(\chi_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Research Reactor</td>
<td>3.0</td>
<td>0.006</td>
<td>0.006</td>
<td>0.024</td>
<td>0.171</td>
<td>0.033</td>
<td>0.240</td>
<td>0.48</td>
</tr>
</tbody>
</table>
8 Summary

In this paper, we have documented 75 problem descriptions with precise results for the critical dimensions, $k_{eff}$ eigenvalue, and some eigenfunction (scalar neutron flux) results for infinite, slab, cylindrical, and spherical geometries for one- and two-energy group, multiple-media, and both isotropic and linearly anisotropic scattering using the listed references. We have not given a complete listing of every referenced result that has been published. Instead, we have included the references that provide both true transport solutions and enough information to reproduce the results. Several other references are included for reference completeness. All test set problems specifications and results are from peer reviewed journals, and have, in many cases, been solved numerically by more than one analytic method. These calculated values for $k_{eff}$ and the scalar neutron flux are believed to be accurate to at least five decimal places. Criticality codes can be verified using these analytic benchmark test problems.

9 ACKNOWLEDGMENTS

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References


APPENDIX A
Derivation of One-, Two-, and Three-Group $k_\infty$

To follow the benchmark referenced literature for the multi-group problems, the lowest energy group is group 1. This notation is the reverse from most nuclear engineering textbooks.

I One-Energy Group Infinite Medium $k_\infty$

For an infinite, isotropic, homogeneous medium, the neutron leakage term, $\Omega \cdot \nabla \Psi = 0$, and the angular and scalar neutron flux is constant everywhere. Integrating the one-energy group infinite medium form of the transport equation over angle produces:

$$\Sigma_t \phi = \Sigma_s \phi + \frac{\nu \Sigma_f \phi}{k_\infty}$$

where $\phi$ is the scalar neutron flux. The equation can be directly solved for $k_\infty$.

$$k_\infty = \frac{\nu \Sigma_f}{\Sigma_t - \Sigma_s}$$

or, in terms of mean number of secondaries, $c$:

$$k_\infty = c \left[ \frac{\nu \Sigma_f \Sigma_t}{(\Sigma_t - \Sigma_s)(\Sigma_s + \nu \Sigma_f)} \right]$$

II Two-Energy Group Infinite Medium $k_\infty$

The two-group infinite medium form of the neutron transport equation reduces to:

$$\Sigma_2 \phi_2 = \Sigma_{2s} \phi_2 + \Sigma_{2t} \phi_1 + \frac{\chi_2}{k_\infty} [\nu_2 \Sigma_{2f} \phi_2 + \nu_1 \Sigma_{1f} \phi_1]$$

$$\Sigma_1 \phi_1 = \Sigma_{1s} \phi_1 + \Sigma_{1t} \phi_2 + \frac{\chi_1}{k_\infty} [\nu_1 \Sigma_{1f} \phi_1 + \nu_2 \Sigma_{2f} \phi_2]$$

Rearranging the equations in terms of $\phi_1$ and $\phi_2$: 
This equation can be written in matrix form as:

\[
\begin{bmatrix}
- \left( \Sigma_{21s} + \chi_{22} \nu_1 \Sigma_{1f} \right) & \left( \Sigma_2 - \Sigma_{22s} - \chi_{22} \nu_2 \Sigma_{2f} \right) \\
\left( \Sigma_1 - \Sigma_{11s} - \chi_{11} \nu_1 \Sigma_{1f} \right) & - \left( \Sigma_{12s} + \chi_{12} \nu_2 \Sigma_{2f} \right)
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]  

(8)

To simplify the matrix elements, it is useful to define a total removal cross section, \(\Sigma_{g}\), for each energy group \(g\) as the difference between the total cross section and in-group scattering or:

\[
\Sigma_{g}^{rem} = \Sigma_g - \Sigma_{22s}
\]

(9)

\[
\Sigma_{1}^{rem} = \Sigma_1 - \Sigma_{11s}
\]

(10)

Setting the determinant of this matrix equal to zero will give an equation that can be solved for \(k_{\infty}\). One solution is \(k_{\infty} = 0\). The other solution is:

\[
k_{\infty} = \chi_1 \left( \nu_2 \Sigma_{2f} \Sigma_{21s} + \Sigma_{2}^{rem} \nu_1 \Sigma_{1f} \right) + \chi_2 \left( \nu_1 \Sigma_{1f} \Sigma_{12s} + \Sigma_{1}^{rem} \nu_2 \Sigma_{2f} \right)
\]

\[
\Sigma_{1}^{rem} \Sigma_{2}^{rem} - \Sigma_{12s} \Sigma_{21s}
\]

(11)

If there is no thermal upscattering, the equation reduces to:

\[
k_{\infty} = \chi_1 \left( \frac{\Sigma_{1}^{rem} \nu_1 \Sigma_{1f}}{\Sigma_{1}^{rem}} \right) + \chi_2 \left( \frac{\nu_1 \Sigma_{1f} \Sigma_{12s}}{\Sigma_{1}^{rem}} + \frac{\nu_2 \Sigma_{2f}}{\Sigma_{2}^{rem}} \right)
\]

(12)

To obtain the flux ratio, equations 23 and 24 are added to eliminate \(\chi_1\) and \(\chi_2\) to give:

\[
\begin{bmatrix}
\Sigma_{1}^{rem} - \Sigma_{21s} - \frac{\nu_1 \Sigma_{1f}}{k_{\infty}} \\
\Sigma_{2}^{rem} - \Sigma_{12s} - \frac{\nu_2 \Sigma_{2f}}{k_{\infty}}
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(13)

where \(\chi_1 + \chi_2 = 1\).

Solving for \(\phi_2/\phi_1\):
Criticality code verification

If there is no thermal upscattering, the equation reduces to:

\[
\frac{\phi_2}{\phi_1} = \frac{\Gamma_{\text{rem}} - \Sigma_{2i} - \nu_1 \Sigma_{1f}}{\Gamma_{\text{rem}} + \Sigma_{12s}}
\]  

(14)

III Three-Energy Group Infinite Medium \(k_\infty\) (a)

To make the three-energy group problem simpler, the following assumptions are made:

- No thermal upscattering from group \(j\) to group \(i\), \(j < i\), \(\Gamma_{ij} = 0\)
- No fission neutrons are born in the lowest energy group, i.e. \(\chi_1 = 0\)

The neutron transport equation can be written as:

\[
\Sigma_3 \phi_3 = \Sigma_{33s} \phi_3 + \frac{\chi_3}{k_\infty} [\nu_3 \Sigma_{3f} \phi_3 + \nu_2 \Sigma_{2f} \phi_2 + \nu_1 \Sigma_{1f} \phi_1]
\]  

(16)

\[
\Sigma_2 \phi_2 = \Sigma_{22s} \phi_2 + \Sigma_{23s} \phi_3 + \frac{\chi_2}{k_\infty} [\nu_3 \Sigma_{3f} \phi_3 + \nu_2 \Sigma_{2f} \phi_2 + \nu_1 \Sigma_{1f} \phi_1]
\]  

(17)

\[
\Sigma_1 \phi_1 = \Sigma_{11s} \phi_1 + \Sigma_{12s} \phi_2 + \Sigma_{13s} \phi_2
\]  

(18)

Rearranging the equations in terms of \(\phi_1\), \(\phi_2\) and \(\phi_3\):

\[
\left[ \Sigma_3 - \Sigma_{33s} - \frac{\chi_3}{k_\infty} \nu_3 \Sigma_{3f} \right] \phi_3 - \left[ \frac{\chi_3}{k_\infty} \nu_2 \Sigma_{2f} \right] \phi_2 - \left[ \frac{\chi_3}{k_\infty} \nu_1 \Sigma_{1f} \right] \phi_1 = 0
\]  

(19)

\[
\left[ \Sigma_2 - \Sigma_{22s} - \frac{\chi_2}{k_\infty} \nu_2 \Sigma_{2f} \right] \phi_2 - \left[ \frac{\chi_2}{k_\infty} \nu_1 \Sigma_{1f} \right] \phi_1 = 0
\]  

(20)

\[-\Sigma_{13s} \phi_3 - \Sigma_{12s} \phi_2 + \left[ \Sigma_1 - \Sigma_{11s} \right] \phi_1 = 0
\]  

(21)

This equation can be written in matrix form as:

\[
\begin{bmatrix}
\Sigma_3 - \Sigma_{33s} - \frac{\chi_3}{k_\infty} \nu_3 \Sigma_{3f} \\
- \Sigma_{23s} + \frac{\chi_2}{k_\infty} \nu_2 \Sigma_{2f} \\
- \Sigma_{13s}
\end{bmatrix}
\begin{bmatrix}
\phi_3 \\
- \Sigma_2 + \frac{\chi_2}{k_\infty} \nu_2 \Sigma_{2f} \\
- \Sigma_{12s}
\end{bmatrix}
- \begin{bmatrix}
\frac{\chi_3}{k_\infty} \nu_2 \Sigma_{2f} \\
\frac{\chi_2}{k_\infty} \nu_1 \Sigma_{1f} \\
\frac{\chi_2}{k_\infty} \nu_1 \Sigma_{1f}
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]  

(22)
Using the total removal cross sections defined in equations 26 and 27, the determinant of the matrix then becomes:

\[
\begin{bmatrix}
\left(\Sigma_3^{rem} - \frac{X_3}{k_{\infty}} \nu_3 \Sigma_3 f\right) & - \left(\frac{X_3}{k_{\infty}} \nu_2 \Sigma_2 f\right) & - \left(\frac{X_3}{k_{\infty}} \nu_1 \Sigma_1 f\right) \\
\left(\Sigma_3^{rem} + \frac{X_2}{k_{\infty}} \nu_3 \Sigma_3 f\right) & \left(\Sigma_2^{rem} - \frac{X_2}{k_{\infty}} \nu_3 \Sigma_3 f\right) & - \left(\frac{X_2}{k_{\infty}} \nu_1 \Sigma_1 f\right) \\
-\Sigma_{13s} & -\Sigma_{12s} & \Sigma_1^{rem}
\end{bmatrix}
\]  

(.23)

If we multiply the second line by \(\chi_3\), multiply the first line by \(\chi_2\), and subtract the results, and multiply the first line by \(k_{\infty}\), the determinant becomes:

\[
\begin{bmatrix}
(\Sigma_3^{rem} k_{\infty} - \chi_3 \nu_3 \Sigma_3 f) & - (\chi_3 \nu_2 \Sigma_2 f) & - (\chi_3 \nu_1 \Sigma_1 f) \\
- (\chi_3 \Sigma_{23s} + \chi_2 \Sigma_3^{rem}) & (\chi_3 \Sigma_2^{rem}) & 0 \\
\Sigma_{13s} & \Sigma_{12s} & \Sigma_1^{rem}
\end{bmatrix}
\]  

(.24)

Two of the \(k_{\infty}\) solutions are zero. The other solution is:

\[
k_{\infty} = \frac{(\chi_3 \Sigma_{23s} + \chi_2 \Sigma_3^{rem}) (\nu_1 \Sigma_{1f} \Sigma_{13s} + \nu_2 \Sigma_2 f \Sigma_1^{rem}) + \chi_3 \Sigma_2^{rem} (\nu_1 \Sigma_{1f} \Sigma_{13s} + \nu_3 \Sigma_3 f \Sigma_1^{rem})}{\Sigma_1^{rem} \Sigma_2^{rem} \Sigma_3^{rem}}
\]  

(.25)

**IV Three-Energy Group Infinite Medium \(k_{\infty} (b)\)**

An alternative method for solving the three-group \(k_{\infty}\) problem \(^{47}\) is to rearrange the three-group transport equations in equations 33, 34, 35.

\[
(\Sigma_3 - \Sigma_{33s}) \phi_3 = \frac{X_3}{k_{\infty}} [\nu_3 \Sigma_3 f \phi_3 + \nu_2 \Sigma_2 f \phi_2 + \nu_1 \Sigma_1 f \phi_1]
\]  

(.26)

\[
(\Sigma_2 - \Sigma_{22s}) \phi_2 = \Sigma_{23s} \phi_3 + \frac{X_2}{k_{\infty}} [\nu_3 \Sigma_3 f \phi_3 + \nu_2 \Sigma_2 f \phi_2 + \nu_1 \Sigma_1 f \phi_1]
\]  

(.27)

\[
(\Sigma_{1} - \Sigma_{11s}) \phi_1 = \Sigma_{12s} \phi_2 + \Sigma_{13s} \phi_3
\]  

(.28)

Divide each equation by \(\phi_3\) and define:

\[
\phi_{23} = \frac{\phi_2}{\phi_3}
\]  

(.29)

\[
\phi_{13} = \frac{\phi_1}{\phi_3}
\]  

(.30)
\[ \Sigma_i^{rem} = \Sigma_i - \Sigma_{iss} \]  \hfill (31)

The result gives:

\[ \Sigma_3^{rem} = \frac{X_3}{k_\infty} \left[ \nu_3 \Sigma_{3f} + \nu_2 \Sigma_{2f} \phi_{23} + \nu_1 \Sigma_{1f} \phi_{13} \right] \]  \hfill (32)

\[ \phi_{23} \Sigma_2^{rem} = \Sigma_{2s} + \frac{X_2}{k_\infty} \left[ \nu_3 \Sigma_{3f} + \nu_2 \Sigma_{2f} \phi_{23} + \nu_1 \Sigma_{1f} \phi_{13} \right] \]  \hfill (33)

\[ \phi_{13} \Sigma_1^{rem} = \Sigma_{1s} + \Sigma_{12s} \phi_{23} \]  \hfill (34)

If we divide equations 50 and 51 by \( \nu_3 \Sigma_{3f} \) and define:

\[ f_{23} = \frac{\nu_2 \Sigma_{2f}}{\nu_3 \Sigma_{3f}} \phi_{23} \]  \hfill (35)

\[ f_{13} = \frac{\nu_1 \Sigma_{1f}}{\nu_3 \Sigma_{3f}} \phi_{13} \]  \hfill (36)

Then we get:

\[ \frac{\Sigma_3^{rem}}{X_3 \nu_3 \Sigma_{3f}} = \frac{1}{k_\infty} \left[ 1 + f_{23} + f_{13} \right] \]  \hfill (37)

\[ \frac{1}{X_2 \nu_3 \Sigma_{3f}} \left[ \Sigma_2^{rem} \phi_{23} - \Sigma_{2s} \right] = \frac{1}{k_\infty} \left[ 1 + f_{23} + f_{13} \right] \]  \hfill (38)

\[ \phi_{13} \Sigma_1^{rem} = \Sigma_{1s} + \Sigma_{12s} \phi_{23} \]  \hfill (39)

If we substitute equation 56 into 57 and rearranging,

\[ \frac{\Sigma_2^{rem}}{\Sigma_3^{rem}} \phi_{23} = \frac{\Sigma_{2s}}{\Sigma_3^{rem}} + \frac{X_2}{X_3} \]  \hfill (40)

\[ \frac{\Sigma_3^{rem}}{\Sigma_3^{rem}} \phi_{13} = \frac{\Sigma_{1s}}{\Sigma_3^{rem}} + \frac{\Sigma_{12s}}{\Sigma_3^{rem}} \phi_{23} \]  \hfill (41)

\[ k_\infty = \frac{X_3 \nu_3 \Sigma_{3f}}{\Sigma_3^{rem}} \left[ 1 + f_{23} + f_{13} \right] \]  \hfill (42)

Equations 57, 58, and 59 give the flux ratios and \( k_\infty \). Tables 43, 44, and 45 give the cross sections used for the three-group problem.\(^{[47]}\)

There are numerous cross sections involved in these equations, implying that there are numerous arbitrary choices we can make that will yield solutions to these equations. We show one set of cross sections that will satisfy a set of chosen conditions.\(^{[47]}\)

If we make our basic choices as:
\[ k_\infty = 1.600 \]

\[ \chi_3 = 0.96, \chi_2 = 0.04, \chi_1 = 0.0 \]

- 5\% of fission production occurs in group 3
- 20\% of fission production occurs in group 2
- 75\% of fission production occurs in group 1

With these choices and the definitions of \( f_{23} \) and \( f_{13} \), we get:

\[
f_{23} = \frac{\nu_2 \Sigma_{2f} \phi_2}{\nu_3 \Sigma_{3f} \phi_3} = 4 \quad (43)
\]

\[
f_{13} = \frac{\nu_1 \Sigma_{1f} \phi_1}{\nu_3 \Sigma_{3f} \phi_3} = 15 \quad (44)
\]

Using this gives:

\[
\frac{\Sigma_{3em}}{\nu_3 \Sigma_{3f}} = \frac{\chi_3}{k_\infty} \left[ 1 + f_{23} + f_{13} \right] \quad (45)
\]

\[
\frac{\Sigma_{3em}}{\nu_3 \Sigma_{3f}} = 12.0 \quad (46)
\]

We can now make more arbitrary choices. If we choose:

\[ \nu_3 = 3.0, \Sigma_{3f} = 0.006 \]

\[ \nu_2 = 2.5, \Sigma_{2f} = 0.060 \]

\[ \nu_1 = 2.0, \Sigma_{1f} = 0.900 \]

Then we get:

\[
\phi_{23} = 0.480 \quad (47)
\]

\[
\phi_{13} = 0.150 \quad (48)
\]

making \( \Sigma_{3em} = 0.216 \) from equation 63. If we make more choices:

\[
\Sigma_{33s} = 0.024 \quad (49)
\]

\[
\Sigma_{3e} = 0.006 \quad (50)
\]

\[
\Sigma_{13s} = 0.033 \quad (51)
\]
making $\Sigma_3=0.240$ and $\Sigma_{23s}=0.171$. Using equation 57, $\Sigma_{2em}^e=0.375$. This result now gives:

$$\Sigma_{22s} = 0.600$$

$$\Sigma_{2c} = 0.040$$

making $\Sigma_2=0.975$ and $\Sigma_{12s}=0.275$. Using equation 58, $\Sigma_{1em}^e=1.10$. One last arbitrary choice is:

$$\Sigma_{11s} = 2.00$$

making $\Sigma_1=3.10$ and $\Sigma_{1c}=0.20$.

V General Multigroup Infinite Medium $k_\infty$

More than three-group $k_\infty$ derivations have been done (see reference [53]). A general multigroup $k_\infty$ derivation is included in this section for completeness.[54]

Given

$$\overline{\Sigma_l} \overline{\phi} = \overline{\Sigma_s} \overline{\phi} + \overline{\chi} \nu \overline{\Sigma_f} \overline{\phi}$$

where:

- $\overline{\Sigma_l} = GxG$ matrix
- $\overline{\Sigma_s} = GxG$ matrix
- $\overline{\chi} = Gx1$ vector
- $\nu \overline{\Sigma_f} = 1xG$ vector
- $\overline{\phi} = Gx1$ vector
- $k = scalar$

then:

$$\left( \overline{\Sigma_l} - \overline{\Sigma_s} \right) \overline{\phi} = \frac{1}{k} \overline{\chi} \nu \overline{\Sigma_f} \overline{\phi}$$

$$\overline{\phi} = \frac{1}{k} \overline{\chi} \nu \overline{\Sigma_f} \overline{\phi}$$

$$\nu \overline{\Sigma_f} \overline{\phi} = \frac{1}{k} \nu \overline{\Sigma_f} \left( \overline{\Sigma_l} - \overline{\Sigma_s} \right)^{-1} \overline{\chi} \nu \overline{\Sigma_f} \overline{\phi}$$
Since $\nu \Sigma f \phi$ is a scalar, it can be cancelled out and we get the following explicit result:

$$k = \nu \Sigma f \left( \overline{\Sigma_t} - \overline{\Sigma_s} \right)^{-1} \chi \quad (59)$$

The right hand side of this equation is a scalar, equal to $k$. Only one matrix inversion is necessary.